Algebra Prelim, January 8, 2019

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

(1) Let $M_{n\times n}(K)$ be the vector space of $n \times n$ matrices over a field K, and let $I_n \in M_{n\times n}(K)$ denote the $n \times n$ identity matrix.

a) Show that the trace map $tr : M_{n\times n}(K) \to K$, $A \to tr(A)$ is K-linear and satisfies $tr(AB) = tr(BA)$.

b) Show there exist no matrices $A, B \in M_{n\times n}(\mathbb{Q})$ such that $AB - BA = I_n$.

c) Find $A, B \in M_{2\times 2}(\mathbb{F}_2)$ such that $AB - BA = I_2$.

(2) Let $A \in M_{3\times 3}(\mathbb{Q})$ have characteristic polynomial $\chi_A(t) = t^3 + 3t^2 + 2t$. Determine the rank of A.

(3) Let G be a p-group. Suppose that G acts on a finite set X such that $p \nmid |X|$. Show that this action has a fixed point.

(4) Consider the symmetric group S_5.

a) Show that there are exactly 20 distinct 3-cycles in S_5.

b) Show that the 3-Sylow subgroups and the 5-Sylow subgroups of S_5 are contained in the alternating group A_5.

c) Determine the number of 3-Sylow subgroups and the number of 5-Sylow subgroups in S_5.

(5) Let A be a commutative ring, and let $P \subset A$ be a prime ideal. For ideals $I, J \subset A$ show that if $I \cap J \subseteq P$ then $I \subseteq P$ or $J \subseteq P$.

(6) Let R and S be integral domains and let $\phi : R \to S$ be a surjective ring homomorphism (in particular, $\phi(1_R) = 1_S$). Prove or find a counterexample to each of the following:

a) If R is a PID then S is a PID.

b) If R is a UFD then S is a UFD.

(7) Let $K = \mathbb{F}_3(t)$, the field of rational functions over $\mathbb{F}_3$. Find a polynomial $p(x) \in K[x]$ which is irreducible but not separable.

(8) a) Find a Galois extension $\mathbb{Q} \subset K$ with $Gal(K/\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$.

b) Find a Galois extension $\mathbb{Q} \subset L$ with $Gal(L/\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

(9) Let $F \subset K$ be a Galois extension with finite Galois group G. Suppose that K is the splitting field of $f(x) \in F[x]$, and that $f(x)$ is the minimal polynomial of $a \in K$. Show:

$$f(x) = \prod_{\delta \in G} (x - \delta(a)).$$

Missing hypothesis: $\delta (a)\neq \gamma (a)$ for all $\delta \neq \gamma \in G$