Algebra Prelim, January 10, 2019

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

(1) Let U and V be finite-dimensional K-vector spaces and $T : U \to V$ be a surjective linear map. Show that there is a subspace $W \subseteq U$ such that the restriction $T|_W \colon W \to V$ is an isomorphism of K-vector spaces.

(2) a) Let φ be an endomorphism on a K-vector space V. Set $U = ker f(\varphi)$, where f is a polynomial with coefficients in K. Show that U is a φ-invariant subspace of V.

b) Let φ be an endomorphism on an $\mathbb{R}$-vector space V whose dimension is an odd number. Argue that V has a one-dimensional φ-invariant subspace.

(3) Let G be a group acting on a set X, and let $N \trianglelefteq G$ be a normal subgroup.

a) State the definition of the kernel of a group action.

b) Let $g, h \in G$ and $a, b, x \in X$. Show that if $h(a) = x$ and $h(b) = g(x)$, then $(h^{-1}gh)(a) = b$.

c) Prove that, if the action of G on X is 2-transitive, and if N is not contained in the kernel of this action, then the action of N on X is transitive.

(4) Let p be an odd prime and let G be a group of order $2^n p$. Let H be a Sylow $2-$subgroup of G. Assume that H is a normal subgroup and that $H \cong (\mathbb{Z}/2\mathbb{Z})^n$. Prove that, if p does not divide $2^n - 1$, then G has a nontrivial center.

(5) Let F be a field and $R = F[x, x^2y, \ldots, x^{n+1}y^n, \ldots] \subset F[x, y]$ be the $F-$subalgebra generated by the monomials of the form $x^{n+1}y^n$ for all $n \in \mathbb{N}$.

a) Show that the field of quotients of R is equal to the field of quotients of $F[x, y]$.

b) Show that R contains an infinite ascending chain of ideals $I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n \subsetneq \cdots$.

(6) Let R be an integral domain, and suppose that every decreasing chain of ideals

$$I_1 \supsetneq I_2 \supsetneq I_3 \supsetneq \cdots$$

is finite in length. Show that R is a field.

(7) Let $K \subseteq L$ be a field extension.

a) Show that $\alpha \in L$ is algebraic over K if and only if $K(\alpha)$ is finite dimensional as a K-vector space.
b) Use part (a) to show that, if $\alpha \in L$ is algebraic over K, then $\beta \in L$ is algebraic over K if and only if β is algebraic over $K(\alpha)$.
c) Use parts (a) and (b) to show that the set of elements of L that are algebraic over K is a field.

(8) Let ζ_n denote a primitive n-th root of unity. Find all subfields of $\mathbb{Q}(\zeta_8)$ and $\mathbb{Q}(\zeta_{12})$. Justify your answer.

(9) Let $\mathbb{F}_q$ denote the field with q elements. How many monic irreducible polynomials of degree 2 are in $\mathbb{F}_q[x]$? Justify your answer.