Algebra Prelim, January 20, 2021

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote the set of integers, rational numbers, real numbers, and complex numbers, respectively.

(1) Let A be an $n \times n$ matrix of rank $n-1$ over a field K.

a) For $k > 0$ let r_k be the rank of the matrix A^k. What are the possible values of r_k?

b) Suppose that $A^\ell = 0$ for some $\ell > 0$, show that $A^k = 0$ for $k \geq n$.

(2) Let W be a subspace of $\mathbb{R}^n$ and let $W^\perp = \{v \in \mathbb{R}^n \mid v \cdot w = 0 \text{ for all } w \in W\}$. Prove that $\mathbb{R}^n = W \oplus W^\perp$.

(3) Let G be a group of order 60 whose Sylow 3-subgroup is normal. Prove that its Sylow 5-subgroup is also normal.

(4) Let a finite group G act on a finite set A. Suppose that this action is faithful (recall that this means that the kernel of the homomorphism from G to $\mathrm{Sym}(A)$ induced by this action is trivial) and transitive (recall that this means that for all $a, b \in A$, there exists $g \in G$ such that $g(a) = b$). For $a \in A$, let G_a denote the stabilizer of a in G.

a) For $a \in A$ and $\sigma \in G$, prove that $G_{\sigma(a)} = \sigma G_a \sigma^{-1}$.

b) For $a \in A$, prove that $\cap_{\sigma \in G} \sigma G_a \sigma^{-1} = \{\mathrm{id}\}$.

c) Suppose that G is abelian. Prove that $|G| = |A|$.

(5) Let R be a finite (not necessarily commutative) ring with multiplicative identity 1_R. Prove that if $a \in R$ is nonzero and is not a zero divisor, then a is a unit in R.

(6) Let R be the quotient ring $\mathbb{C}[x, y, z, w]/(xy - zw)$.

a) Show that R is an integral domain.

b) Show that R is not a UFD.

(7) Let $\mathbb{F}_q$ be the finite field of cardinality q. Let $f \in \mathbb{F}_q[x]$ be an irreducible polynomial of degree n and let α be a root of f in an extension field of $\mathbb{F}_q$.

a) Find $[\mathbb{F}_q(\alpha) : \mathbb{F}_q]$.

b) Prove that $\alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{n-1}}$ are n distinct roots of f.

c) Argue that $\mathbb{F}_q(\alpha)/\mathbb{F}_q$ is a Galois extension.

(8) Let $z \in \mathbb{C}$ be a primitive n^{th} root of 1, $n \geq 3$. Let $y = z + z^{-1}$ and let $K = \mathbb{Q}(y)$.

a) Find (with proof) $[K : \mathbb{Q}]$.

b) Find $\mathbb{Q}(z) \cap \mathbb{R}$ and $[\mathbb{Q}(z) \cap \mathbb{R} : \mathbb{Q}]$.

(9) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 4 whose splitting field K over $\mathbb{Q}$ has Galois group $G = S_4$. Let θ be a root of $f(x)$.

a) Prove that $\mathbb{Q}(\theta)$ is a field extension of $\mathbb{Q}$ of degree 4.

b) Prove that there are no intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\theta)$.