# Algebra Prelim, January 7, 2022

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

**Good luck!**

**(1)** Let $n \in \mathbb{N}$ and $\mathbb{F}$ be a field. Let $A, B \in \mathbb{F}^{n\times n}$.

a) Suppose $A$ is invertible. Show that $AB$ and $BA$ have the same minimal polynomial. [Hint: One option is to consider $f(AB)A$ for $f \in \mathbb{F}[x]$.]

b) Give an example showing that $AB$ and $BA$ do not have the same minimal polynomial if neither matrix is invertible.

**(2)** Let $V$ be a vector space over a field $\mathbb{K}$ and let $W \subseteq V$ be a subspace. The dual space of $V$, written $V^*$, is defined to be the set of all linear maps $f : V \to \mathbb{K}$. Similarly, $W^*$ is the set of all linear maps $f : W \to \mathbb{K}$. Define a map $\pi : V^* \to W^*$ by $\pi(f) = f|_W$, where $f|_W$ denotes the restriction of $f$ to $W$. Prove that $\pi$ is a surjective map.

**(3)** Let $p, q$ be primes such that $2 < p < q$. Let $G$ be a group of order $2pq$.

a) Show that $G$ is not simple.

b) Show that $G$ is solvable.

[Fun fact: 2022 is of the form $2pq$.]

**(4)** a) How many conjugates does $(12)(3456)$ have in $S_7$?

b) How many elements in $S_7$ commute with $(12)(3456)$? Describe these elements.

**(5)** Let $\mathbb{K}$ be a field. Recall that a $\mathbb{K}$-algebra automorphism of the ring $\mathbb{K}[x]$ is a ring automorphism $\phi : \mathbb{K}[x] \to \mathbb{K}[x]$ such that $\phi(\alpha) = \alpha$ for every element $\alpha \in \mathbb{K}$. Let $\mathrm{Aut}(\mathbb{K}[x] \mid \mathbb{K})$ denote the group of $\mathbb{K}$-algebra automorphisms of $\mathbb{K}[x]$.

a) Show that any $\phi \in \mathrm{Aut}(\mathbb{K}[x] \mid \mathbb{K})$ is determined by the image $\phi(x)$ of $x \in \mathbb{K}[x]$, and that $\phi(x) = \alpha x + \beta$ for some $\alpha \in \mathbb{K} \setminus \{0\}$ and $\beta \in \mathbb{K}$.

b) Show that any $\alpha \in \mathbb{K} \setminus \{0\}$ and $\beta \in \mathbb{K}$ determine a unique element $\phi_{\alpha,\beta} \in \mathrm{Aut}(\mathbb{K}[x] \mid \mathbb{K})$.

c) Compute $\phi_{\alpha,\beta}^{-1}$ for $\alpha \in \mathbb{K} \setminus \{0\}$ and $\beta \in \mathbb{K}$.

d) Show that elements of the form $\phi_{1,\beta}$ with $\beta \in \mathbb{K}$ form a normal subgroup of $\mathrm{Aut}(\mathbb{K}[x] \mid \mathbb{K})$.

**(6)** Let $R$ be a commutative ring with identity. Denote its group of units by $R^*$. Let $I \subset R$ be an ideal. Show that the following are equivalent.

a) $I$ is the unique maximal ideal of $R$.

b) $R \setminus I = R^*$.

c) $I$ is a maximal ideal and $1 + a \in R^*$ for all $a \in I$.

[You may use without proof that every proper ideal is contained in a maximal ideal.]

**(7)** Let $\mathbb{K} \,|\, \mathbb{F}$ be a finite extension of fields and assume that $\mathbb{F}$ has characteristic $p > 0$. Recall that $\mathbb{K}^p = \{a^p \mid a \in \mathbb{K}\}$.

a) Prove that $\mathbb{K}^p$ is a subfield of $\mathbb{K}$.

b) Prove that $[\mathbb{K} : \mathbb{F}] = [\mathbb{K}^p : \mathbb{F}^p]$.

c) Prove that $[\mathbb{F} : \mathbb{F}^p] = [\mathbb{K} : \mathbb{K}^p]$.

**(8)** Let $f = x^4 - 2 \in \mathbb{Q}[x]$.

a) Show that $\mathbb{Q}[x]/(f)$ is a field.

b) Let $\mathbb{E}$ be a splitting field of $f$ over $\mathbb{Q}$. Show that $[\mathbb{E} : \mathbb{Q}] = 8$.

c) Determine the number of field homomorphisms from $\mathbb{E}$ to $\mathbb{C}$.

d) Let $G$ be the Galois group of $f$ over $\mathbb{Q}$ and $\mathcal{X}$ be the set of roots of $f$ in $\mathbb{E}$. Show that for every root $\alpha \in \mathcal{X}$ there exists a $\sigma \in G \setminus \{\mathrm{id}_{\mathbb{E}}\}$ such that $\sigma(\alpha) = \alpha$.

[Hint: One option is to consider the action of $G$ on $\mathcal{X}$.]

**(9)** Let $\mathbb{F}$ be a field of characteristic zero and let $\mathbb{F}(\alpha, \beta) \,|\, \mathbb{F}$ be a finite Galois extension. Assume furthermore that $\mathbb{F}(\alpha) \,|\, \mathbb{F}$ and $\mathbb{F}(\beta) \,|\, \mathbb{F}$ are also Galois extensions and that $\mathbb{F}(\alpha) \cap \mathbb{F}(\beta) = \mathbb{F}$. Set $G = \mathrm{Gal}(\mathbb{F}(\alpha, \beta) \,|\, \mathbb{F}(\alpha + \beta))$. Let $\sigma \in G$. Show the following.

a) $\sigma(\alpha) - \alpha = \beta - \sigma(\beta)$, and this element is in $\mathbb{F}$.

b) $\sigma^m(\alpha) = m\sigma(\alpha) - (m-1)\alpha$ for all $m \in \mathbb{N}$.

[Hint: Induct on $m$ and make sure to cover $m = 2$.]

c) $\mathbb{F}(\alpha, \beta) = \mathbb{F}(\alpha + \beta)$.

[Hint: Use that $G$ is finite.]

Make sure to explain where the characteristic of $\mathbb{F}$ is needed.