Algebra Prelim, January 6, 2023

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even if you did not successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**
- **All problems carry the same weight.**

Good luck!

(1) Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ endowed with an inner product $\langle \cdot , \cdot \rangle$. Suppose $\{v_1, \ldots, v_r\}$ is an orthonormal basis of V. Show that every $v \in V$ satisfies

$$v = \sum_{i=1}^{r} \langle v, v_i \rangle v_i.$$

(2) Let $p \geq 3$ be a prime number and $k \leq p - 2$. Show that the identity matrix I_k is the only matrix $A \in \mathrm{GL}_k(\mathbb{Q})$ such that $A^p = I_k$.
[Hint: Consider the minimal polynomial of A.]

(3) Let G be a group acting on a set X, where $|X| > 1$. Suppose the following:

i) G acts transitively on X (that is, for all $x, y \in X$ there exists $g \in G$ such that $g \cdot x = y$).
ii) Every $g \in G$ has a fixed point in X (that is, $g \cdot x = x$ for some $x \in X$).

Denote by G_x the stabilizer of $x \in X$ in G.

a) Let $x \in X$. Show that $G_x \neq G$.
b) Fix $a \in X$ and set $H = G_a$. Show that $G = \bigcup_{g \in G} gHg^{-1}$.

(4) Consider the symmetric group S_8.

a) Determine the number of 5-cycles in S_8.
b) Let $\sigma \in S_8$ be an element of order 15. Determine the cycle type of σ.
c) Determine the number of elements of order 15 in S_8.

(5) Let R be a UFD and $P \subset R$ be a prime ideal. Suppose the only prime ideals contained in P are 0 and P. Show that P is a principal ideal.

(6) Consider the ring of Gaussian integers $R = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$. Let

$$J = \{25x + (7 + i)y \mid x, y \in \mathbb{Z}\}.$$

a) Show that if $iJ \subseteq J$ then J is an ideal in R.

One can easily see that $iJ \subseteq J$ is indeed true and thus J is an ideal. Use this fact without proof for the following parts.

b) Show that the ring homomorphism $\phi : \mathbb{Z} \longrightarrow R/J,\ a \longmapsto a + J$ is surjective.
[Hint: Find first a pre-image of $i + J$.]

c) Determine $\ker \phi$.

d) Determine the order of the multiplicative group $(R/J)^*$.

e) Show that the group of units $(R/J)^*$ is cyclic.

(7) Let $f = x^7 + x + 1 \in \mathbb{F}_2[x]$.

a) Show that f has no roots in $\mathbb{F}_2$, $\mathbb{F}_4$, and $\mathbb{F}_8$.
[Hint: For $\mathbb{F}_4$ and $\mathbb{F}_8$ make use of the multiplicative order of their elements.]

b) Show that f is irreducible.
[Hint: Consider the degree of an irreducible factor.]

(8) Let $f = x^5 + 5 \in \mathbb{Q}[x]$ and $E \subseteq \mathbb{C}$ be the splitting field of f over $\mathbb{Q}$. Let $G := \mathrm{Gal}(E \mid \mathbb{Q})$ and set $\alpha = \sqrt[5]{-5}$ and $\zeta = e^{\frac{2\pi i}{5}}$.

a) Show that $E = \mathbb{Q}(\alpha, \zeta)$.

b) Determine $[E : \mathbb{Q}]$.

c) Find a subfield F of E that is not Galois over $\mathbb{Q}$.

d) Show that G is not abelian.

e) Show that G contains a nontrivial normal subgroup.

(9) Let $E \mid F$ be a Galois extension of degree $132 = 2^2 \cdot 3 \cdot 11$. Show that there exists a field $\widehat{F}$ such that $F \subsetneq \widehat{F} \subsetneq E$ and $\widehat{F}|F$ is Galois.