Algebra Prelim, January 5, 2024

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even if you did not successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**
- **The points shown below give an indication about the weight of each problem (the weight may not split evenly across the individual parts of a problem). Pass/fail is based on the overall performance: completeness, rigor, and presentation.**

Good luck!

Notation: $F^{n\times m}$ denotes the set of all $n \times m$-matrices with entries in the field F. For any ring R we denote by R^* its group of units. $\mathrm{GL}_n(F)$ denotes the general linear group of degree n, that is the group of units in the matrix ring $F^{n\times n}$.

(1) (10 points) Let F be an algebraically closed field and $A \in F^{n\times n}$. Show

$$A^n = 0 \Longleftrightarrow I_n - \lambda A \in \mathrm{GL}_n(F) \text{ for all } \lambda \in F.$$

Make sure to explain where you need that F is algebraically closed.

(2) (10 points) Let F be a field and $n_1, n_2 \in \mathbb{N}$. Set $n = n_1 + n_2$. Let $A_i \in F^{n_i\times n_i}$ and consider the block diagonal matrix

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \in F^{n\times n}.$$

Suppose A is diagonalizable, say $S^{-1}AS = D$, where D is diagonal and $S \in \mathrm{GL}_n(F)$. Write

$$S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \quad \text{where } S_i \in F^{n_i\times n}.$$

Note that the partitioning of the matrices allows for block matrix multiplication.

a) Show that the nonzero columns of S_i are eigenvectors of A_i.

b) Show that A_1 and A_2 are diagonalizable.

(3) (10 points)

a) Let G_1, G_2, G be abelian groups with $+$ as operation. Suppose there exist group homomorphisms $\phi_i : G_i \longrightarrow G$. Show that the map $(g_1, g_2) \to \phi_1(g_1) + \phi_2(g_2)$ defines a group homomorphism from $G_1 \times G_2$ to G.

b) Show that for every $n \in \mathbb{N}$ there exists an element of order n in the group $(\mathbb{Q}/\mathbb{Z}, +)$.

c) Show that for every finitely generated abelian group $(G, +)$ there exists a non-trivial group homomorphism $\psi : G \longrightarrow \mathbb{Q}/\mathbb{Z}$.

[Hint: You may use the Fundamental Theorem of Finitely Generated Abelian Groups.]

(4) (10 points) Let G be a finite group and $\mathcal{X} = \{H \leq G\}$, that is, $\mathcal{X}$ is the set of all subgroups of G. Consider the action

$$G \times \mathcal{X} \longrightarrow \mathcal{X},\ (g, H) \longmapsto gHg^{-1}$$

and denote by $\mathcal{O}_H$ the orbit of $H \in \mathcal{X}$. (You do not need to show that the above is a group action.) Show the following.

a) For any $H \in \mathcal{X}$ we have $|\mathcal{O}_H| = 1 \iff H \trianglelefteq G$.

b) Let p be a prime and G be a nontrivial p-group. Let $n = |\mathcal{X}|$ and m be the number of normal subgroups of G. Show that $p \mid (n - m)$.

(5) (15 points) Let R be a commutative ring that contains a field F as a subring. Suppose R is a 2-dimensional F-vector space. Show the following:

a) There exists an element $a \in R$ such that $R = F[a]$.

b) There exists a monic polynomial $f \in F[x]$ of degree 2 such that $R \cong F[x]/(f)$.

c) R is either a field or isomorphic to one of the rings $F \times F$ and $F[x]/(x^2)$.

d) $F \times F$ and $F[x]/(x^2)$ are not isomorphic.

(6) (10 points) Let R be a principal ideal domain and let F be the field of fractions of R. Let $c \in F$. Prove that every finitely generated ideal in $R[c]$ is a principal ideal.

(7) (10 points) Let $\mathbb{F}_q$ be the finite field with q elements. Let ℓ be a prime number such that $\ell \nmid q$. Suppose that for some $a \in \mathbb{F}_q$ the polynomial $x^\ell - a \in \mathbb{F}_q[x]$ is irreducible. Prove that $q \equiv 1 \bmod \ell$.

[Hint: For one possible solution, consider the group homomorphism $\mathbb{F}_q^* \to \mathbb{F}_q^*,\ x \mapsto x^\ell$.]

(8) (10 points) Let $p \geq 3$ be prime and $\zeta \in \mathbb{C}$ be a primitive p-th root of unity.

a) Show that $[\mathbb{Q}(\zeta) : \mathbb{Q}(\zeta + \zeta^{-1})] = 2$.

b) Show that $\mathbb{Q}(\zeta + \zeta^{-1}) \mid \mathbb{Q}$ is Galois with cyclic Galois group of order $(p-1)/2$.

(9) (10 points) Let E be a subfield of $\mathbb{C}$ such that $E \mid \mathbb{Q}$ is Galois and $G := \mathrm{Gal}(E \mid \mathbb{Q})$ is cyclic of order 4, say $G = \langle \sigma \rangle$.

a) Show that E is closed under complex conjugation.

b) Show that $i \notin E$.

[Hint: Consider the fixed field $\mathrm{Fix}(\langle \sigma^2 \rangle)$.]