ALGEBRA PRELIM, JANUARY 2025

- **Provide proofs of all statements, citing theorems that may be needed.**
- **Do as many problems as possible and present your solutions as carefully as possible.**
- **All problems carry the same weight, but the individual parts of a problem may have different weights.**

(1) Let V be a finite dimensional vector space and let I_1, $I_2 \subseteq V$ be linearly independent sets with $|I_1| < |I_2|$. Prove that there exists $v \in I_2$ such that $I_1 \cup \{v\}$ is linearly independent.

(2) Let $A \in M_n(\mathbb{C})$ be an $n \times n$ matrix with entries in $\mathbb{C}$. Suppose that A is invertible and A^{2024} is diagonalizable. Let $\lambda_1, \cdots, \lambda_m$ be all the distinct eigenvalues of A^{2024} in $\mathbb{C}$. Let $p(x) \in \mathbb{C}[x]$ be the minimal polynomial of A.

(a) Prove that $p(x)$ divides $(x^{2024} - \lambda_1)(x^{2024} - \lambda_2) \cdots (x^{2024} - \lambda_m)$.

(b) Prove that $p(x)$ has no repeated roots.

(c) Prove that A is diagonalizable.

(3) Let H, K be subgroups of a group G. Suppose that H is normal in G. Consider the subsets

$$HK = \{hk : h \in H, k \in K\}, \quad KH = \{kh : k \in K, h \in H\}.$$

(a) Prove that HK is a subgroup of G.

(b) Prove that $HK = KH$.

(4) Let p be a prime and let G be a group of order p^n for some integer $n \geq 1$.

(a) Show that the center of G is nontrivial.

(b) Show that G is solvable.

(5) Let R be a commutative ring with unit such that every $x \in R$ satisfies $x^n = x$ for some $n > 1$. Show that every prime ideal of R is a maximal ideal.

(6) Consider the ring $R = \mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$.

(a) Find all units in R.

(b) Determine if the element 2 is irreducible in R.

(c) Determine if the element 2 is prime in R. (Equivalently, if $x, y \in R$ and 2 divides xy, does it follow that 2 divides either x or y?)

(d) Determine if R is a UFD.

(7) In the ring of Gaussian integers $\mathbb{Z}[i]$, the ideal $(1 - i)$ is prime. What is the order of the field $\mathbb{Z}[i]/(1 - i)$?

(8) Let $K \subset \mathbb{C}$ be the splitting field for the polynomial $x^3 - 2$ over $\mathbb{Q}$.

(a) Determine $[K : \mathbb{Q}]$, and give a $\mathbb{Q}$-basis for K.

(b) Identify the Galois group $G = G(K/\mathbb{Q})$.

(c) Find all subfields of K. For each of them, determine if it is Galois over $\mathbb{Q}$ or not.

(9) Let $K \subset L$ be a Galois field extension of degree 351. Show that there exists a field K' with $K \subsetneq K' \subsetneq L$ such that K'/K is Galois.