

Algebra Prelim

June 2000

Provide proofs for all statements, citing any theorems that may be needed.

If necessary, you may use the results from other parts of this test, even though you may not have successfully proved them.

In the following, \mathbb{Q} denotes the field of rational numbers and \mathbb{Z} denotes the ring of integers.

All 10 questions carry equal weight in determining your grade.

1. Let G be a group.
 - (a) Define what is meant by the derived group G' of G .
 - (b) Show that G' is a normal subgroup of G .
 - (c) Prove that the quotient G/G' is abelian.
 - (d) Prove that any group intermediate H satisfying $G' < H < G$ is necessarily a normal subgroup of G .
2. Let $n \in \mathbb{Z}$ be a positive integer and set $\mathbb{Z}_n = \mathbb{Z}/(n)$. Show that the following are equivalent:
 - (a) n is prime.
 - (b) \mathbb{Z}_n is an integral domain.
 - (c) \mathbb{Z}_n is a field.
3. Prove that any group of order 312 is not simple.
4. Prove that the center $Z(G)$ of a group of order 231 contains an 11-Sylow subgroup of G .

Give an example of such a group of order 231 for which the center has order exactly 11.
5. Let $p(x) = x^3 - 3x - 1$ in $\mathbb{Q}[x]$.

- (a) Show that $K = \mathbb{Q}[x]/(p(x))$ is a field and find a basis for K over \mathbb{Q} .
- (b) Determine a splitting field of $p(x)$ over \mathbb{Q} and find the Galois group of $p(x)$ over \mathbb{Q} . You may use that the formula for the discriminant of a cubic $x^3 + 3ax + b$ is $-27(4a^3 + b^2)$.
6. Let $F \subset K$ be fields and let $\alpha, \beta \in K$ be algebraic over F .
- (a) Prove that $\alpha + \beta$ is algebraic over F .
- (b) Prove that $\alpha\beta$ is also algebraic over F .
- (c) Explain why the set of elements of K which are algebraic over F form a field.
7. Let M be the \mathbb{Z} -module \mathbb{Z}^3 consisting of columns of height three with integer entries (i.e. the M is a free module of rank three over \mathbb{Z}). Let $f : M \rightarrow M$ be defined by $f(v) = Av$ where $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.
- (a) What is the kernel of f ? Is f injective?
- (b) Is f surjective?
- (c) Determine the number of elements in $\mathbb{Z}^3/Im(f)$ where $Im(f)$ is the image of f , i.e. $\{f(v)|v \in \mathbb{Z}^3\}$.
- (d) What would change in your answers if the same matrix is used to define a map from \mathbb{Q}^3 to itself?
8. Explicitly construct the splitting field of $x^3 - x - 1$ over $GF(3)$ and determine its Galois group over $GF(3)$.
9. Give examples (with justification) of the following:
- (a) An integral domain which is not a UFD.
- (b) A division ring which is not a field.
10. Consider the matrix $B = \begin{pmatrix} 2 & 1 & 0 \\ -2 & 5 & 0 \\ 1 & 2 & 1 \end{pmatrix}$. Calculate the characteristic polynomial of B and its eigenvalues. Use your work to diagonalize B .