Algebra Prelim

June 2002

Provide proofs for all statements, citing any theorems that may be needed. If necessary, you may use the results from other parts of this test, even though you may not have successfully proved them.

In the following, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Z}$ denotes the ring of integers and $\mathbb{C}$ the field of complex numbers.

1. Let G be a 3-group acting on a set S of 24 elements. Suppose that there is an element $s_1 \in S$ such that $g \cdot s_1 = s_1 \;\; \forall g \in G$.

   Prove that there is $s_2 \in S$ distinct from s_1 such that $g \cdot s_2 = s_2 \;\; \forall g \in G$.

2. Prove that the polynomial $f(X) = X^3 + 6X + 2$ is irreducible over $\mathbb{Q}$. Let θ be a root of $f(X)$ in $\mathbb{C}$. Determine (with proof) the minimum polynomial of $-\theta$ over $\mathbb{Q}$. Determine (also with proof) the minimum polynomial for $1 - \theta$.

3. Let G be a group and let H_1, H_2 be two distinct maximal subgroups of G. If K is a normal subgroup of H_1 as well as H_2, then argue that K is a normal subgroup of G.

   Discuss if the above conclusion can hold if $H_1 = H_2$.

4. Let I be the ideal $(2 - X, X + 3)$ in $\mathbb{Z}[X]$.

   (a) Prove that I is a prime ideal.

   (b) Prove that I is not a principal ideal.

   (c) Determine $I \cap \mathbb{Z}$.

5. Let A, B be $n \times n$ matrices over a field k with $n \geq 1$ and let their columns be denoted as $A_1, \cdots, A_n$ and $B_1, \cdots, B_n$ respectively. Assume that they have the same null space, i.e.

$$AX = 0 \text{ iff } BX = 0 \quad \forall X \in k^n.$$

   (a) Construct an example of a pair of such matrices over a field of your choice, satisfying:
      - A and B have dimension at least two.
      - A and B are not scalar multiples of each other.
      - A, B satisfy the stated condition.

   (b) If $1 \leq r \leq n$ prove that $A_1, \cdots, A_r$ are linearly independent iff $B_1, \cdots, B_r$ are linearly independent.

   (c) Prove that A, B have the same rank.

6. Let F_p be a finite field with p elements and let $f \in F_p[X]$ be an irreducible polynomial of degree n.

   Argue that if α is a root of f in some extension field L, then $\alpha, \alpha^p, \cdots, \alpha^{p^{n-1}}$ are all the **distinct** roots of f in L.

   Using this or otherwise, argue that $F_p(\alpha)$ is a Galois extension of F_p and determine its Galois group.

7. Let p, q be distinct primes in $\mathbb{Q}$. Let $L = \mathbb{Q}\,(\sqrt{p}, \sqrt{q})$.

   (a) Prove that $[L : \mathbb{Q}\,] = 4$.

   (b) Determine the Galois group of L over $\mathbb{Q}$.

   (c) Explicitly find an element $\theta \in L$ such that $L = \mathbb{Q}\,(\theta)$.

8. Let $g(X)$ be an irreducible monic polynomial of degree $n > 1$ over $\mathbb{Q}$.

   (a) Prove that there is a polynomial $h_1(Y) \in \mathbb{Q}\,[Y]$ of degree n such that $h_1(2X)$ is divisible by $g(X)$. Explain if possible how to find it concretely using the given $g(X)$.

   (b) Prove that there is a polynomial $h_2(Y) \in \mathbb{Q}\,[Y]$ of degree n such that $h_1(X^2)$ is divisible by $g(X)$. Explain if possible how to find it concretely using the given $g(X)$.

   (c) As an illustration, or practice, do the above exercise for the concrete polynomial $X^3 - 2$.