Algebra Prelim

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1. Let $K := \mathbb{Z}/3\mathbb{Z}$ and let α be a root of $X^3 + 2X + 1 \in K[X]$ in some extension field of K. Show that $X^3 + 2X + 1$ is irreducible over K and compute the multiplicative inverse of $\alpha + 1$ in $K[\alpha]$.

2. Let $\mathbb{R}$ denote the field of real numbers. Determine up to isomorphism all rings S such that there is a surjective ring homomorphism $\mathbb{R}[X]/(X^2 + 1)(X - 2)(X - 3) \to S$.

3. A commutative ring R is a PIR (*principal ideal ring*) if each ideal of R is a principal ideal. Let I be an ideal of the ring R. If R is a PIR, then show that R/I is a PIR.
If R/I is a PIR, does it follow that R is a PIR? Why, or why not?

4. Let K be any field and let $a_1, \ldots, a_n \in K$ be n different elements. Let $b_1, \ldots, b_n \in K$ be any elements. Show that there is a unique polynomial $f \in K[X]$ of degree $< n$ such that $f(a_i) = b_i$ for all $i = 1, \ldots, n$.

5. Factor the following polynomials into irreducible factors in $\mathbb{Z}[X]$:

$4X^2 + 10X + 6$

$X^8 - 4$

$X^{16} - 15X + 35.$

6. Let $f := X^{12} - 1 \in \mathbb{Q}[X]$.
(a) Compute the Galois group $G(f, \mathbb{Q})$.
(b) Let E be the splitting field of f over $\mathbb{Q}$. Determine all subfields of E.

7. Let G be a group of order 200. Show that G has a Sylow 5-subgroup that is normal and abelian.

8. Decide whether there is field K such that the following groups are isomorphic to the multiplicative group of K. Either find K or explain why there is no such K.
(a) $\mathbb{Z}_3 \times \mathbb{Z}_8$
(b) $\mathbb{Z}_2 \times \mathbb{Z}_4$

9. Let A be an $m \times n$ matrix with entries in a field k. Show that the rank of A is m if and only if the matrix equation $A\vec{x} = \vec{b}$ has a solution for each vector $\vec{b} \in k^{(m)}$.

10. Let G be a finite group.
(a) Write down the class equation for G.
(b) Use (a) to prove that if p is a prime number and $|G| = p^a$ for some $a \geq 1$, then G has a nontrivial center.