Algebra Prelim

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1. Let $g, h \in G$ where G is a group. Suppose that $gh = hg$ and that $|g| = 10$ and that $|h| = 12$. Prove that $|gh| \geq 30$.

2. Prove that the symmetric group S_5 is generated by the two permutations $(12) \circ (345)$ and (12345).

3. Let V be a vector space of dimension 8 over some field k. Let $T : V \to V$ be a linear transformations such that $dim(ker(T)) = 1$. Then:

   (a) Determine the possibilities for the dimensions of $T(V)$, $T^2(V) = T(T(V))$, $T^3(V) = T(T^2(V)), \ldots$.

   (b) Argue that if $T^n = 0$ for some $n \geq 1$ then $n \geq 8$ and in fact that in this case $T^8 = 0$.

4. (a) Let G be a finite group. Suppose that G has exactly 5 Sylow p-subgroups for some prime p. Explain why G has an element of order 5 and an element of order 2.

   (b) Now let G be a finite group such that G has exactly $1 + 2^n$ Sylow p-subgroups for some prime p and some $n \geq 1$. Explain why G has an element of order q for each prime q that divides $1 + 2^n$ and also has an element of order 2.

5. Let $\Re$ be the field of real numbers. Consider the ring homomorphism $\phi : \Re[x] \to M_2(\Re)$ (where $M_2(\Re)$ is the ring of 2×2 matrices over $\Re$) that maps $r \in \Re$ to $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ and that maps x to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then:

   (a) Find the kernel of ϕ.

   (b) Find the dimension of $\Re[x]/ker(\phi)$ as a vector space over $\Re$

   (c) Is $\Re[x]/ker(\phi)$ an integral domain or not? Explain your answer.

6. Let R be a commutative ring. Suppose that for some $a, b \in R$ we have $as + bt = 1$ for some $s, t \in R$. Then:

   (a) Prove that the function $r \mapsto (r + (a), r + (b))$ from R to $R/(a) \times R/(b)$ is surjective and that its kernel is $(a) \cap (b)$

   (b) Show that $(a) \cap (b) = (ab)$

   (c) Consider $17, 33 \in \mathbb{Z}$. Find all integers n such that n has a remainder of 3 when divided by 13 and a remainder of 5 when divided by 33.

7. Let $\mathbb{C}(x)$ be the field of fractions of $\mathbb{C}[x]$ where $\mathbb{C}$ is the field of complex numbers. Let $\zeta_6 \in \mathbb{C}$ be a primitive 6-th root of unity. Consider the unique homomorphism $\sigma : \mathbb{C}(x) \to \mathbb{C}(x)$ over $\mathbb{C}$ such that $\sigma(x) = \zeta_6 x$. Then:

   (a) Find the order of σ as an element of the group of automorphism of $\mathbb{C}(x)$, i.e. of $Aut(\mathbb{C}(x))$.

   (b) Argue that $\mathbb{C}(x^6)$ is in the fixed field of σ.

   (c) Show that $\mathbb{C}(x^6) \subset \mathbb{C}(x)$ is a Galois extension and give its Galois group.

   (d) Note that $\mathbb{C}(x^6) \subset \mathbb{C}(x^3) \subset \mathbb{C}(x)$. Using this, determine $Gal(\mathbb{C}(x)/\mathbb{C}(x^3))$ as a subgroup of $Gal(\mathbb{C}(x)/\mathbb{C}(x^6))$.

8. Let k be a finite field. Let $f(x) \in k[x]$ be such that $f(0) \neq 0$ and such that $gcd(f(x), f'(x)) = 1$. Prove that for some $n \geq 1$, $f(x)$ divides $x^n - 1$ in $k[x]$.