Department of Mathematics
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Algebra Prelim

Provide proofs for all statements, citing any theorems that may be needed.

In the following, $\mathbb{Z}$ denotes the ring of integers. We write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$.

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1. Let C be a cyclic group of order 6. Find necessary and sufficient conditions on a group G in order that $C \times G$ be a cyclic group.

2. Let R be a non-zero ring containing $1 \neq 0$ such that the function $r \mapsto r^2$ from R to R is a homomorphism of rings. Prove that R is a commutative ring of characteristic 2.

3. Prove that the group of automorphisms of the abelian group $\mathbb{Z}_3 \times \mathbf{Z}$ has 12 elements.

4. Precisely state the Sylow Theorems.

   Let G be a group of order 105.

   Let $n_p(G)$ denote the number of p-Sylow subgroups of G as usual.

   For each prime p less than 10 determine possible values of $n_p(G)$ and the corresponding estimate of number of elements of order p in G.

   Using these calculations or otherwise prove that G cannot be simple.

5. Let $k \subset L$ be a Galois extension where $|Gal(L/k)| = 75$.

   (a) Prove that there is a unique field F with $k \subset F \subset L$ such that $[F : k] = 3$.

   (b) Prove that the field F constructed above is a Galois extension of k.

Please turn over.

6. Let V be a 3-dimensional vector space over a field k. Assume that you have three linear transformations f, g, h from V to k with the following properties.

   - There is $u \in V$ such that $f(u) = 1, g(u) = h(u) = 0$.
   - There is $v \in V$ such that $g(v) = h(v) = 1$.
   - There is $w \in V$ such that $g(w) = 2, h(w) = 3$.

   Define a linear transformation from V to k^3 by $L(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}$.

   Answer the following:

   (a) Determine if L is surjective.

   (b) Determine if L is injective.

   (c) Determine $Ker(f) \bigcap Ker(g) \bigcap Ker(h)$.

7. Consider the polynomial $X^5 - 2$ over $\mathbb{Z}_{11}$ and let $R = \mathbb{Z}_{11}[X]/(X^5 - 2)$. As usual, identify $\mathbb{Z}_{11}$ with its image in R.

   Define the ring homomorphism $\sigma : R \to R$ by $\sigma(t) = t^{11}$ for all $t \in R$.

   (a) Determine the order of σ (i.e. the smallest n such that $\sigma^n = Id$).

   (b) Using the above or otherwise, argue that R is a field.

   (c) Determine the Galois group of R over $\mathbb{Z}_{11}$.

8. Give the precise definition of a prime ideal and a maximal ideal in a commutative ring T with $1 \neq 0$.

   Let $R = \mathbb{Z}[X]$.

   (a) Determine with proof if $I = (X^4 + 2, X^2 + 1) \subset R$ is prime or maximal or neither.

   (b) Determine with proof if $J = (X^4 + X, X^2 + 1) \subset R$ is prime or maximal or neither.