Algebra Prelim

May 28, 2008

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

(1) Let V be a vector space of dimension 10 over some field K. Argue that there is an automorphism f on V such that $f^{10} := f \circ f \circ \ldots \circ f$ (ten f's) is the identity map on V and such that f^i is not the identity for $1 \leq i \leq 9$.

(2) Let $(G, \cdot)$ be a group with identity element e. Suppose that $a \neq e$ is an element of G such that $a^6 = a^{10} = e$. Determine the order of a.

(3) For a group G denote its center by $Z(G)$.
Prove or disprove (by giving an example) the statement:

For each group G and each subgroup H of G one has $Z(H) = H \cap Z(G)$.

(4) Show that there is no simple group of order 56.

(5) Show that every ideal of $\mathbb{Z} \times \mathbb{Z}$ is principal.

(6) Let $\mathbb{F}_3$ be the field with 3 elements. Compute the monic greatest common divisor d of the polynomials

$$f = x^3 + 2, \quad g = 2x^2 + x + 1 \in \mathbb{F}_3[x].$$

Also give a Bezout equation for d, that is, find polynomials $a, b \in \mathbb{F}_3[x]$ such that $d = af + bg$.

(7) Factor the following (possibly irreducible) polynomials into their irreducible factors in the given polynomial ring.

a) $f := 2x^7 - 8x^6 + 4x^3 - 12x + 4 \in \mathbb{Z}[x]$.

b) $g := x^3 + 1 \in \mathbb{F}_3[x]$.

c) $h := x^7 - 1 \in \mathbb{Q}[x]$.

d) $k := 3y^3x^3 - 4yx^3 + 6y^2x^2 + 2x + 1 \in \mathbb{Q}[x, y]$.

(8) Let $f = x^3 - 3x + 4 \in \mathbb{Q}[x]$ and let K be the splitting field of f inside $\mathbb{C}$.

a) Determine the Galois group of f over $\mathbb{Q}$ up to isomorphism.

b) Show that $K = \mathbb{Q}(a, i)$, where a is any root of f in $\mathbb{C}$.

(9) Let F be a field of order $q := p^r$, where p is a prime number and $r \in \mathbb{N} := \{1, 2, 3, \ldots\}$, and let $f \in F[x]$ be an irreducible polynomial of degree $n > 1$.

a) Argue that $K := F[x]/(f)$ is a splitting field of f over F.

b) Determine all roots of f in K explicitly (remember, the elements of K are cosets).

c) Determine the size of the splitting field of $(x^2 + 1)(x^2 + 2x + 2) \in \mathbb{F}_3[x]$.