Algebra Prelim

June 1, 2011

- **Provide proofs for unsupported statements and cite supporting theorems**
- **Do as many problems as you can while giving careful solutions**

Good Luck!

1. Let G be a finite simple group where $|G| = 210$. Find the number of elements of G having order 7.

2. Let G be a group such that $G/Z(G)$ is a cyclic group. Prove that G is abelian (here $Z(G)$ denotes the center of G).

3. Let V be a vector space over a field F and let $T : V \to V$ be a linear map. Let $Fix(T) = \{v | T(v) = v\}$. Then $Fix(T)$ is a subspace of V (you do not need show this). Now suppose that $T \circ T = T$. Prove the following:

   (a) $Fix(T) = Im(T)$, where $Im(T)$ denotes the image of T.

   (b) $Ker(T) \cap Im(T) = \{0\}$.

   (c) $Ker(T) + Im(T) = V$, where $+$ here denotes the usual sum of subspaces.

4. Let $A \in \mathcal{M}_{n\times n}(\mathbf{C})$ be a Hermitian matrix. Prove or disprove (with a counterexample) the following statements:

   (a) $det(A) \in \mathbf{R}$.

   (b) $|det(A)| = 1$.

   (c) If A has exactly one eigenvalue then A is a real matrix.

   (d) If $v = (v_1, \ldots, v_n)^T$ is an eigenvector of A then $\overline{v} = (\overline{v_1}, \ldots, \overline{v_n})^T$ is also an eigenvector for A (here $\overline{v_i}$ denotes the complex conjugate of v_i).

5. Let $f(x) = x^{12} - 1 \in \mathbf{Q}[x]$ and let $L \subset \mathbf{C}$ be a splitting field of $f(x)$ over $\mathbf{Q}$. Then:

   (a) Determine the isomorphism type of $\mathrm{Gal}(L/\mathbf{Q})$

   (b) Give an explicit description of all the subfields of L (by giving generators for each of these over $\mathbf{Q}$).

(please turn over)

6. Let K be a field and let $f(x) \in K[x]$. Then let L be a splitting field of $f(x)$ over K. Assume that $f(x)$ has degree n for some $n \geq 1$ and that $f(x)$ has n distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ in L. Suppose that for every i, j with $1 \leq i, j \leq n$ there is an automorphism σ of L over K such that $\sigma(\alpha_i) = \alpha_j$. Prove that $f(x)$ is irreducible in $K[x]$.

7. Show that the following polynomials are irreducible elements of the given integral domains.

   (a) $f(x) = 2x^4 + 120x^3 - 30x^2 + 18x + 60 \in \mathbf{Q}[x]$

   (b) $f(x) = 5x^3 - 2x^2 - 3x + 105 \in \mathbf{Q}[x]$

   (c) $f(x, y) = x^2y + xy^2 - x - y + 1 \in \mathbf{Q}[x, y]$

8. Let K be a field that is a subfield of the integral domain S. So S is a vector space over K. If $\dim_K(S) < \infty$ show that S is a field.

9. Prove that there is a surjective ring homomorphism $\mathbf{Z} \to \mathbf{Z}/(2) \times \mathbf{Z}/(3)$ and that there is not a surjective ring homomorphism $\mathbf{Z} \to \mathbf{Z}/(6) \times \mathbf{Z}/(10)$.

10. Let $k \subset L$ be a Galois extension where $G = \text{Gal}(L/k)$ and where $|G| = 150$. Now suppose we have $k \subset K \subset L$ where $k \subset K$ is a Galois extension of degree 2. Prove that there is exactly one K' with $k \subset K' \subset L$ where $[L : K'] = 25$.
Hint. This is essentially a Sylow subgroup problem.