Algebra Prelim

May 30, 2012

- Provide proofs for all statements, citing theorems that may be needed.
- If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.
- Do as many problems as you can and present your solutions as carefully as possible.

Good Luck!

(1) Let K be a field and V be a finite-dimensional vector space over K. Let $F : V \to V$ be a diagonalizable linear map on V. Suppose that for all eigenvectors $v, w \in V$ of F the sum $v + w$ is either zero or also an eigenvector of F. Show that $F = \lambda \cdot \mathrm{id}_V$.
[Hint: Show first that F has at most one eigenvalue.]

(2) Let V be the vector space of all polynomial functions $p : \mathbb{R} \to \mathbb{R}$ such that $p(x) = c_0 + c_1 x + c_2 x^2$ for some $c_i \in \mathbb{R}$ (that is, all polynomial functions of degree at most 2). Define the following three linear functionals on V:

$$f_1(p) = \int_0^1 p(x)dx, \quad f_2(p) = \int_0^2 p(x)dx, \quad f_3(p) = \int_0^{-1} p(x)dx.$$

Show that $\{f_1,\ f_2,\ f_3\}$ is a basis of the dual space V^*.

(3) Let $G = \langle x \rangle$ be a cyclic group of order n. For each integer $a \in \mathbb{Z}$ define the map

$$\sigma_a : G \longrightarrow G, \quad g \longmapsto g^a$$

Observe that σ_a is a group homomorphism (you need not prove this). Show the following:
a) σ_a is an automorphism of G if and only $(a, n) = 1$.
b) Every group homomorphism $\varphi : G \to G$ is equal to σ_a for some a.
c) $\sigma_a = \sigma_b$ if and only if $a \equiv b \bmod n$.

(4) Let G be a group of order 30.
a) Show that G has a normal subgroup of order 3 or a normal subgroup of order 5.
b) Show that if G has a normal subgroup of order 3 then it also has a normal subgroup of order 5.
[Hint: You may want to consider a quotient group.]

(please turn over)

(5) Let $n \in \mathbb{N}_{\geq 2}$ and $m \in \mathbb{N}_{\geq 1}$. Show that the polynomial

$$\sum_{i=1}^{n} x_i^m - 1 \in \mathbb{Q}[x_1, \ldots, x_n]$$

is irreducible.
[Hint: Induct on $n \geq 2$ and recall that $\mathbb{Q}[x_1, \ldots, x_n] = \mathbb{Q}[x_1, \ldots, x_{n-1}][x_n]$. There is hardly any computation involved in this problem.]

(6) Let F be a field and $f \in F[x]$ be a polynomial of degree n. Suppose that f has n distinct roots in F. Show that the rings $F[x]/(f)$ and $\underbrace{F \times \ldots \times F}_{n \text{ factors}}$ are isomorphic.

(7) Let $f = (x^5+8x^3+18)(x^5-1) \in \mathbb{Q}[x]$. Let K be a splitting field of f and let $[K : \mathbb{Q}] = n$. Show that $20 \mid n$.

(8) Let $f = x^9 - 1 \in \mathbb{Q}[x]$, and let K be a splitting field of f over $\mathbb{Q}$.
a) Determine the number of subfields of K.
b) Determine the number of subfields of K that are Galois over $\mathbb{Q}$.

(9) Let p be a prime number. Consider the finite fields $\mathbb{F}_{p^4}$ and $\mathbb{F}_p$. We may assume that $F_p \subseteq \mathbb{F}_{p^4}$.
a) Show that there exist exactly $p^4 - p^2$ elements $a \in \mathbb{F}_{p^4}$ such that $\mathbb{F}_{p^4} = \mathbb{F}_p(a)$.
[Hint: Consider the subfields of $\mathbb{F}_{p^4}$.]
b) Show that there exist exactly $\frac{1}{4}(p^4 - p^2)$ monic irreducible polynomials of degree 4 in $\mathbb{F}_p[x]$.