Algebra Prelim

June 5, 2013

- Provide proofs for all statements, citing theorems that may be needed.
- If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.
- Do as many problems as you can and present your solutions as carefully as possible.

Good luck!

1. Let W be a subspace of $V = M_n(\mathbb{C})$, the $\mathbb{C}$-vector space of all $n \times n$ complex matrices. Assume that every nonzero matrix in W is invertible. Prove that $\dim_{\mathbb{C}} W \leq 1$.

2. Let K be a field with 8 elements, say $K = \mathbb{Z}_2[x]/(x^3 + x + 1)$.
   (*a*) Prove that the Frobenius map, defined by $\varphi(\alpha) = \alpha^2$ for any $\alpha \in K$, is a linear transformation of K, when K is viewed as a vector space over $\mathbb{Z}_2$.
   (*b*) Choose a basis for the $\mathbb{Z}_2$-vector space K and write the matrix representation of φ with respect to this basis.
   (*c*) Determine the eigenvalues and the eigenvectors of φ.
   (**Hint:** you have to perform your calculations in a suitable field extension of $\mathbb{Z}_2$ in order to find all the eigenvalues and eigenvectors of φ).

3. Let G be a group of order 48. Show that G must contain a normal subgroup of order 8 or 16. (**Hint:** If $n_2(G) > 1$ let G act on $\mathrm{Syl}_2(G)$ via conjugation.)

4. Let p be prime number and let G be a group of order p^n. Let H be a non-trivial normal subgroup of G and let $Z(G)$ denote the center of G. Show that $H \cap Z(G)$ is non-trivial.

5. Let $n, m \geq 1$ be positive integers with greatest common divisor d. Show that the ideal of $\mathbb{Q}[x]$ generated by $x^m - 1$ and $x^n - 1$ is principal and generated by $x^d - 1$.

6. Let R be an integral domain with fraction field K.
   (*a*) Assume in addition that R is a unique factorization domain. Suppose that the monic polynomial
   $$p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in R[x]$$
   has a root $\alpha \in K$. Show that $\alpha \in R$.
   (*b*) Use part (*a*) to argue that the subring $R = k[t^2, t^3]$ of the polynomial ring $k[t]$, where t is an indeterminate over the field k, is not a unique factorization domain. (**Hint:** consider, for example, the polynomial $p(x) = x^2 - t^2 \in R[x]$.)

7. Let E be a field extension of $\mathbb{Z}_p$, where p is a prime, contained in the algebraic closure $\overline{\mathbb{Z}_p}$. Let f be an irreducible polynomial in $\mathbb{Z}_p[x]$ and let $\alpha, \beta \in \overline{\mathbb{Z}_p}$ be roots of f. If $\alpha \in E$, show that $\beta \in E$.

8. Let $f = x^6 + 3 \in \mathbb{Q}[x]$ and let $\alpha \in \mathbb{C}$ denote a 6-th root of -3. Set $\zeta = \frac{1}{2}(1 + \alpha^3) \in \mathbb{C}$.
   (*a*) Show that ζ is a primitive 6-th root of unity and $K = \mathbb{Q}(\alpha)$ is the splitting field of f over $\mathbb{Q}$.
   (*b*) Show that $\mathrm{Gal}(K/\mathbb{Q}) = \{\sigma_0, \ldots, \sigma_5\}$, where $\sigma_i(\alpha) = \zeta^i \alpha$ for $i = 0, \ldots, 5$.
   (*c*) Show that $\sigma_i(\zeta) = \zeta$ for $i = 0, 2, 4$ and $\sigma_i(\zeta) = \zeta^{-1}$ for $i = 1, 3, 5$.
   (*d*) Determine the order of each automorphism σ_i and show that $\mathrm{Gal}(K/\mathbb{Q})$ is not cyclic.