Algebra Prelim

June, 2014

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them. Be sure to refer to such used parts.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

1. Consider a linear transformation T on a vector space V of dimension four over $\mathbb{R}$ the reals. On a basis e_1, e_2, e_3, e_4 of V, the transformation is defined by:

$$T(e_1) = e_2,\ T(e_2) = e_1,\ T(e_3) = 2e_3 + e_4,\ \text{and } T(e_4) = e_3 - 2e_4.$$

(a) Construct the matrix A of the transformation in the given basis.

(b) Determine the characteristic polynomial, the eigenvalues and eigenspaces of A.

(c) Determine the kernel and the image of the transformation defined by the matrix $A^2 - I$ on $\mathbb{R}^4$.

(d) Is A diagonalizable? Would you answer differently, if the field $\mathbb{R}$ is replaced by the field of rationals $\mathbb{Q}$?

2. Let G be a finite group and H a subgroup so that $[G : H] = d$ with $1 < d < |G|$.

(a) Briefly describe the natural homomorphism $\phi : G \to S_d$ where S_d is considered to be the permutation group on the d cosets of H in G.

(b) Prove that if $|G|$ does not divide $d!$, then H has a non-trivial subgroup K such that $K \lhd G$.

(c) Using the above or otherwise show that a group G of order 24 must contain a normal subgroup of order 4 or 8.

3. Let $\phi : \mathbb{Z}[X] \to \mathbb{Z}[X]/(X^2 + 7) = R$ be the natural residue class homomorphism. Let $\phi(X) = x$.

(a) Prove that $1 + x$ is irreducible in R.

(b) Prove that $(1 + x) \subset R$ is not a prime ideal.

(c) Is R a U.F.D.? Why?

(d) Is R a P.I.D.? If not then also present a concrete ideal of R that is not principal.

4. Let K be a field, X, Y, t indeterminates and $\phi : K[X, Y] \to K[t]$ a K-algebra homomorphism. Let $P \subset K[X, Y]$ be the kernel of ϕ.

(a) Prove that P is a prime ideal of $K[X, Y]$.

(b) Assume that the image of ϕ is contained in K. Prove that P is a maximal ideal of $K[X, Y]$.

(c) Let $\phi(X) = t^2 + 2, \phi(Y) = t^3 + 3$. Argue that P is principal and find a generator for P.

5. Determine if the given polynomials are irreducible over the indicated fields.

(a) $f(X) = X^3 + X + 6$ over $\mathbb{Q}$.

(b) $g(Y) = Y^5 + X^2Y^4 - 3XY + X(1+X)$ over $\mathbb{Q}(X)$.

6. Let R be a commutative ring with $1 \neq 0$. Recall that a proper ideal I of R is said to be primary if it satisfies the condition:

$$\text{If } ab \in I \text{ then either } a \in I \text{ or } b \in Rad(I).$$

The ideal $Rad(I)$ is the set of elements x such that $x^n \in I$ for some positive integer n.

(a) Briefly, explain why an ideal I is primary if and only if every zero divisor in the ring R/I is nilpotent.

This form of the condition is often easier to check.

(b) If Q is a primary ideal of R, then prove that $P = Rad(Q)$ is a prime ideal. We may express this by saying Q is P-primary.

(c) Let $A = K[X,Y]$, a polynomial ring in two variables over a field K. Prove that $I = (X+Y, Y^2)$ is primary in A. Identify $Rad(I)$.

7. Let K be a subfield of the reals and $f(X)$ be a monic polynomial of degree $n > 1$ over K.

(a) Let r be the number of real roots of $f(X)$ counted with multiplicity. Show that $n - r$ is even.

(b) Assume that $K = \mathbb{Q}$ and that $f(X)$ is irreducible of degree 3 with exactly one real root. Prove that the Galois group of the splitting field of $f(X)$ over $\mathbb{Q}$ is S_3.

(c) Consider the cubic polynomial $f(X) = X^3 - 3pX + 2p$, where p is a prime number of the form $p = 1 + 3d^2$ for some integer d. Argue that the Galois group of the splitting field of $f(X)$ over $\mathbb{Q}$ is A_3. Where is the primeness of p used?

You may use the formula that the discriminant of $X^3 - 3aX + 2b$ is $108(a^3 - b^2)$.

8. Let $f(X) = (X^3 - 5)(X^5 - 7) \in \mathbb{Q}[X]$, and let K be a splitting field of $f(X)$ over $\mathbb{Q}$. Let $n = [K : \mathbb{Q}]$.

(a) Argue that n is divisible by 15.

(b) Show that K must contain a primitive 15-th root of unity over $\mathbb{Q}$ which satisfies a monic polynomial of degree 8.

(c) Deduce that $n = 120$.

9. Let $f(x) = x^4 + x + 1 \in GF(2)[x]$ be a polynomial in x over the field $GF(2)$ with two elements. Let K be a splitting field of $f(x)$ over $GF(2)$.

(a) Determine $[K : GF(2)]$.

(b) Determine the Galois group of $f(x)$ (i.e. $Gal(K, GF(2))$).

(c) Let $\alpha \in K$ be a root of $f(x)$. Give an explicit representation of all roots of $f(x)$ in K in terms of α.

(d) Determine the smallest number $m \in \mathbb{N}$ such that $f(x)$ divides $(x^m - 1)$ in $GF(2)[x]$.