Algebra Prelim, May 27, 2015

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

(1) Let K be a field of characteristic not equal to 2. Let

$$M := \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \in K^{2\times 2}$$

and consider the linear map

$$\varphi : K^{2\times 2} \longrightarrow K^{2\times 2},\ X \longmapsto MX - XM.$$

a) Find the matrix representation of φ with respect to the standard basis of $K^{2\times 2}$.

b) Find $\ker \varphi$.

[You may want to find first the kernel (null space) of the matrix representation from a). But your final answer to b) needs to be a subspace of $K^{2\times 2}$.]

c) Find all eigenvalues of φ.

d) Show that φ is diagonalizable.

(2) Let K be a field, a an element in a field extension such that a is algebraic over K. Denote by $f \in K[x]$ the minimal polynomial of a over K. Consider the K-vector space $V := K[x]/(f)$ and the K-linear map

$$\varphi : V \longrightarrow V, \quad g + (f) \longmapsto xg + (f).$$

Thus, $\varphi \in \mathrm{End}(V)$. Show that the minimal polynomial of φ is given by f.

(3) Let G be a finite group and $\mathcal{X} = \{H \leq G\}$, that is, $\mathcal{X}$ is the set of all subgroups of G. Consider the action

$$G \times \mathcal{X} \longrightarrow \mathcal{X},\ (g, H) \longmapsto gHg^{-1}$$

and denote by $\mathcal{O}_H$ the orbit of $H \in \mathcal{X}$. Show the following.

a) For any $H \in \mathcal{X}$ we have $|\mathcal{O}_H| = 1 \Longleftrightarrow H \trianglelefteq G$.

b) Let p be a prime and G be a nontrivial p-group. Let $n := |\mathcal{X}|$ and m be the number of normal subgroups of G. Show that $p \mid (n - m)$.

(4) Consider the group $G := (\mathbb{Q}, +)/(\mathbb{Z}, +)$.

a) Let $a, b \in \mathbb{Z}$ with $b \neq 0$ and suppose $\gcd(a, b) = 1$. Show that $\langle \frac{a}{b} + \mathbb{Z} \rangle = \langle \frac{1}{b} + \mathbb{Z} \rangle$ for the cyclic subgroups of G generated by the given elements.

b) Show that for each $n \in \mathbb{N}$ there exists a unique subgroup of order n.

(5) Let K be a field and $f,\ g \in K[x]$. Show that the following two statements are equivalent.
i) There exists a ring homomorphism of the form

$$\varphi : K[x]/(f) \longrightarrow K[x]/(g),\ p + (f) \longmapsto p + (g).$$

ii) g divides f in $K[x]$.

(6) Consider the ring $\mathbb{Z}[i]$ of Gaussian integers, and let f be the ring homomorphism

$$f : \mathbb{Z} \longrightarrow \mathbb{Z}[i]/(3+2i),\ c \longmapsto c + (3+2i).$$

Show the following.

a) f is surjective.

b) $\ker f = 13\mathbb{Z}$.

c) $|\mathbb{Z}[i]/(3+2i)| = 13$.

[Hint: Have in mind that 2 and 3 are relatively prime.]

(7) Let $[K : F] = n$ and let $a \in K$ such that there exist automorphisms $\sigma_1, \dots, \sigma_n \in \mathrm{Aut}(K \mid F)$ with $\sigma_i(a) \neq \sigma_j(a)$ whenever $i \neq j$. Show $K = F(a)$.

(8) Consider the field extension $\mathbb{F}_{5^4} \mid \mathbb{F}_5$.

a) Determine the number of elements $a \in \mathbb{F}_{5^4}$ satisfying $\mathbb{F}_{5^4} = \mathbb{F}_5(a)$.

b) Determine the number of irreducible polynomials of degree 4 in $\mathbb{F}_5[x]$.

(9) Denote by Z_n the cyclic group of order n.

a) Find a field extension $K \mid \mathbb{Q}$ such that $\mathrm{Gal}(K \mid \mathbb{Q}) \cong Z_5$.
[Hint: Start with a primitive 11th root of unity.]

b) Let $L = K(\sqrt{2})$. Argue that $L \mid \mathbb{Q}$ is Galois and determine the cardinality of $\mathrm{Gal}(L \mid \mathbb{Q})$.

c) Give the isomorphism type of the Galois group $\mathrm{Gal}(L \mid \mathbb{Q})$ and describe the automorphisms explicitly.