Algebra Prelim, June 7, 2016

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

(1) In the real vector space of continuous real-valued functions defined on $\mathbb{R}$ consider the functions $p_i, i = 0, 1, 2$, and exp, defined as

$$p_i(x) = x^i, \ \exp(x) = e^x \text{ for all } x \in \mathbb{R}.$$

Set $V := \text{span}_{\mathbb{R}}\{p_0, p_1, p_2, \exp\}$ and consider the endomorphism $\sigma : V \longrightarrow V$ defined as

$$(\sigma f)(x) := f(x-1) \text{ for all } x \in \mathbb{R}.$$

a) Give the matrix representation of σ with respect to the basis $\{p_0, p_1, p_2, \exp\}$. (You need not show that this set is indeed a basis of V.)

b) Determine all eigenvalues and find bases of all eigenspaces of σ.

c) Is σ diagonalizable?

d) Determine the minimal polynomial of σ.

(2) Let V be an n-dimensional vector space over a field K, and let U be a k-dimensional subspace of V. Consider the set

$$M = \{\varphi : V \to V \mid \varphi \text{ is linear and } \varphi(U) \subseteq U\}.$$

a) Argue that M is a K-vector space.

b) Determine the dimension of M.

(3) Let G be a group with center Z. Assume that G/Z is cyclic. Show that G is abelian.

(4) Let G be a finite group, and let p be the smallest prime divisor of the order of G. Suppose H is a subgroup of G with index p. Show that H is a normal subgroup of G.
(Hint: Consider the permutation representation induced by the action of G on the cosets of H by left multiplication.)

(5) Let R, S be commutative rings with identity.

a) Prove that every ideal of the product ring $R \times S$ is of the form $I \times J$, where I is an ideal of R and J is an ideal of S.

b) Describe all prime ideals of $R \times S$ in terms of the ideals of R and S.

(6) Consider the ring $R := \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ differentiable}\}$ and the ideal

$$I := \{f \in R \mid f(2) = f'(2) = 0\}.$$

(You need not show that I is an ideal.)

a) Find a suitable map $R \to \mathbb{R}[X]/(X^2)$ to show that the rings R/I and $\mathbb{R}[X]/(X^2)$ are isomorphic.

b) Show that every ideal of R/I is a principal ideal.

(7) Let $n \in \mathbb{N}$, and let K be a field whose characteristic does not divide n. Consider $f = X^n - c \in K[X]$ for some $c \neq 0$, and let E be a splitting field of f over K. Thus, E contains a primitive n-th root of unity, say ζ. (You need not show this.)

a) Argue, for any root $\alpha \in E$ of f, that $E = K(\zeta, \alpha)$.

b) Suppose $\zeta \in K$. Show that all irreducible factors of f have degree $[E : K]$, and conclude that $[E : K]$ divides n.

c) Assume $\zeta \notin K$. Suppose $n = 2^k$ is a power of 2. Use induction to prove that $[K(\zeta) : K]$ is a power of 2.

d) Suppose n is a power of 2. Use (b) and (c) to show that $[E : K]$ is a power of 2.

(8) Let E be a splitting field of $f = X^6 + 1$ over $\mathbb{Q}$.

a) Describe all automorphisms of E explicitly, and determine the isomorphism type of this automorphism group.

b) Describe all subfields of E by specifying suitable elements that one needs to adjoin to $\mathbb{Q}$.

(9) Let $\alpha := \sqrt{5 + 2\sqrt{6}} \in \mathbb{R}$.

a) Compute the minimal polynomial f of α over $\mathbb{Q}$.

b) Show that f splits into linear factors over $\mathbb{Q}(\alpha)$. (Hint: Check that $\frac{1}{\alpha}$ is a root of f.)

c) Find the isomorphism type of the Galois group of f over $\mathbb{Q}$.

d) How many subfields does $\mathbb{Q}(\alpha)$ have?