Algebra Prelim, May 30, 2018

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

(1) Let V be a finite dimensional vector space over a field F and let $T : V \to V$ be a linear transformation. Assume that $T^2 = T$. Prove the following statements.

a) $\text{im}\,(T) \cap \ker\,(T) = (0)$.

b) $V = \text{im}\,(T) \bigoplus \ker\,(T)$.

c) There exists a basis β of V such that the matrix of T with respect to β is a diagonal matrix where each diagonal entry lies in $\{0, 1\}$.

(2) Let $V \subset \mathbb{R}[x]$ be a vector space of dimension k. We say that a polynomial f vanishes to order n at $a \in \mathbb{R}$ if $f(a) = 0$ and n is the smallest positive integer such that $f^{(n)}(a) \neq 0$.

a) Show that $V_n = \{f \in V | f \text{ vanishes to order } \geq n \text{ at } a\}$ is a subspace of V.

b) Let $a \in \mathbb{R}$. Show that $\dim(V_n) - \dim(V_{n+1})$ is either 0 or 1.

c) Conclude that there are precisely k integers n such that there exists a nonzero $f \in V$ that vanishes to order n at a.

(3) Let p be a prime number and let $G = \mathbb{Z}/p^3\mathbb{Z} \bigoplus \mathbb{Z}/p^5\mathbb{Z} \bigoplus \mathbb{Z}/p^7\mathbb{Z} \bigoplus \mathbb{Z}/p^9\mathbb{Z} \bigoplus \mathbb{Z}/p^{11}\mathbb{Z}$. How many elements in G have order p^8?

(4) Let G be a finite group that acts transitively on a set X with $|X| > 1$. Show that G contains at least one element with no fixed points.

(5) Let G be a finite group with identity element e, and let H, K be cyclic, normal subgroups of G such that $H \cap K = \{e\}$ and $|G| = |H||K|$. Prove the following statements.

a) $hk = kh$ for all $h \in H$ and $k \in K$.

b) If $|H|$ and $|K|$ are relatively prime, then G is cyclic.

(6) Let R be a commutative ring with multiplicative identity. Let I and J be two ideals in R. Use the first isomorphism theorem to prove the following statements.

a) $(I + J)/J \cong I/(I \cap J)$.

b) If $I \subseteq J$, then $(R/I) / (J/I) \cong R/J$.

(7) Let R be a commutative ring with multiplicative identity. Prove that $R[x]$ is a PID if and only if R is a field.

(8) Let p be a prime number with $p \neq 2, 3$. Prove that the degree of the splitting field of $x^{12} - 1$ over $\mathbb{F}_p$ is either 1 or 2. Can you give a rule to determine when the degree is 1 and when the degree is 2?

(9) Let $K \subset \mathbb{C}$ be the splitting field $x^{28} - 1$ over $\mathbb{Q}$.

a) Find the Galois group of K over $\mathbb{Q}$,

b) Find the lattice of all subfields of K. (You do not need to give generators for each field. You need to give the containment relations and the relative degrees of the various field extensions.)