Algebra Prelim, May 31, 2019

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

(1) Let $\mathbb{F}_3$ be the field of order 3. Consider the matrix

$$M = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in \mathbb{F}_3^{3\times 3}.$$

Show that

$$\mathbb{F}_3[M] := \left\{ \sum_{i=0}^{n} a_i M^i \,\middle|\, n \in \mathbb{N}_0, a_i \in \mathbb{F}_3 \right\}$$

is a field of order 27.

(2) Let A be an integral domain which contains a field $K \subset A$ and suppose that A is finite dimensional as a vector space over K. Show that A is a field extension of K.

(3) Let G be a group and let $[G, G]$ be the subgroup of G generated by the set $\{hgh^{-1}g^{-1} \mid h, g \in G\}$.

a) Let A be an abelian group. Show that if $\phi : G \to A$ is a group homomorphism, then $[G, G] \subset Ker(\phi)$.

b) Show that if H is a subgroup of G containing $[G, G]$, then $H \trianglelefteq G$ and G/H is an abelian group.

(4) Let p be a prime and let G be a finite group with $Aut(G) \cong \mathbb{Z}/p\mathbb{Z}$.

a) Show that $Aut(G)$ contains a subgroup isomorphic to $G/Z(G)$.

b) Use (a) to show that G is abelian.

c) Use (b) to show that $p = 2$.
[Hint: Make use of the map $f : G \to G$ given by $f(x) = x^{-1}$.]

(5) Show that the following polynomials are irreducible in the given ring.

a) $f = x^3 + (y^2 - 1)x^2 + 3(y^2 - y)x - 4y + 4 \in \mathbb{Q}[x, y]$.

b) $g = y^3 + xy + x^2(x - 1)^2 \in \mathbb{R}[x, y]$.

c) $h = 5x^4 + 4x^3 - 2x^2 - 3x + 21 \in \mathbb{Q}[x]$.

(6) Consider the ring

$$R := \mathbb{Z}[2i] = \{a + 2bi \mid a, b \in \mathbb{Z}\}.$$

a) Determine the field of fractions, Q, of R inside $\mathbb{C}$.

b) Show that $f = x^2 + 1$ is reducible in $Q[x]$, but irreducible in $R[x]$.

c) Argue that R is not a UFD.

(7) Let $K \mid F$ be a finite, normal field extension and $L \mid K$ be any field extension. Furthermore, let $\varphi : K \longrightarrow L$ be an F-homomorphism. Show that $\varphi(K) \subseteq K$.

(8) Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

a) Show that $K \mid \mathbb{Q}$ is Galois and that $Gal(K \mid \mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

b) Show that $K = \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

c) Find the minimal polynomial of $\sqrt{2} + \sqrt{3}$.

(9) Let K be a field of characteristic $p > 0$, let $a \in K$ and let β be a root of the polynomial $f(x) = x^p - x - a$.

a) Show that $\beta + 1$ is also a root of $f(x)$. Conclude that $K(\beta)$ is a Galois extension of K.

b) Determine the Galois group of this extension. Give explicitly all its elements and give its isomorphism type.