Algebra Prelim, June 04, 2020

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote the set of integers, rational numbers, real numbers, and complex numbers, respectively.

(1) Let K be a field, and let $A \in \mathrm{M}_{n\times n}(K)$ be an $n \times n$ matrix with entries in K. Show that A is invertible if and only if the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution for every vector $\mathbf{b} \in K^n$.

(2) Let A be a square matrix with entries in $\mathbb{Q}$.

a) State the definition of the minimal polynomial of A.

b) Suppose A^3 is the identity matrix. List all options for the minimal polynomial of A. For each polynomial on your list, give an example of some matrix with this minimal polynomial. (Remember to argue why your example has the desired property.)

c) For each polynomial on your list, find a diagonal matrix D with entries in $\mathbb{C}$ such that $A = PDP^{-1}$ for the corresponding matrix A and some invertible matrix P (you DO NOT need to find P).

(3) Let G be a finite group having exactly 2 conjugacy classes. Prove that G has order 2.

(4) Let G be a group of order 105. Let Q, R be Sylow $5, 7$-subgroups, respectively.

a) Prove that at least one of Q and R is normal in G.

b) Prove that G has a cyclic normal subgroup H of order 35.

c) Prove that both Q and R are normal in G.

(5) Let $I = \langle x - 2, x + 3\rangle \subset \mathbb{Z}[x]$.

a) Prove that I is a prime ideal.

b) Prove that I is not a principal ideal.

(6) Let F be a field and let R be the set of 2×2 matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a, b \in F$.

a) Show that R equipped with matrix multiplication and addition is a commutative ring with identity.

b) Show that if $F = \mathbb{R}$ then $R \cong \mathbb{C}$.

c) Is R a field for every field F? Prove or find a counterexample.

(7) Let $\mathbb{F}_q$ be the finite field with q elements and let $\mathbb{F}_q^\times$ denote the multiplicative group of nonzero elements of $\mathbb{F}_q$. Let n be a positive integer and let $d = \gcd(n, q-1)$. Find $|(\mathbb{F}_q^\times)^n|$, the number of nonzero n^{th} powers in $\mathbb{F}_q$.

(8) Let $F \subset K \subset L$ be finite degree field extensions. Determine if each of the following assertions is true or false. If true, explain why; if false, give a counterexample.

a) If L/F is Galois, then so is K/F.

b) If L/F is Galois, then so is L/K.

c) If L/K and K/F are both Galois, then so is L/F.

(9) Let $K \subset \mathbb{C}$ be the splitting field of $x^{24} - 1$ over $\mathbb{Q}$.

a) Find the Galois group of K over $\mathbb{Q}$.

b) How many subfields does K have?

Be sure to justify your answers.