Algebra Prelim, June 4, 2021

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{F}_q$ denote the sets of integers, non-negative integers, rational numbers, real numbers, complex numbers, and the finite field of order q, respectively.

(1) For a real $n \times n$ matrix M and a real number $\lambda \in \mathbb{R}$ we let $\operatorname{eig}(M, \lambda)$ denote the space of vectors $\mathbf{v} \in \mathbb{R}^n$ such that $M\mathbf{v} = \lambda\mathbf{v}$. Find a 4×4 matrix A with

$$\ker(A) = \langle \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rangle, \quad \operatorname{eig}(A, -1) = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rangle, \quad \operatorname{eig}(A, 2) = \langle \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \rangle.$$

(2) Let V be a finite-dimensional vector space, and let $S, T : V \to V$ be linear maps. Prove that $\dim(\ker(S \circ T)) \leq \dim(\ker(S)) + \dim(\ker(T))$.

(3) Let G be a group of order $m > 2$. Show that G has a non-trivial automorphism. It may be helpful to consider the following cases separately.

a) G is non-abelian.

b) G is abelian and contains an element g with order > 2.

c) G is abelian and all non-identity elements of G have order 2.

(4) Recall that a group G is said to be *solvable* if there is a finite sequence of subgroups:

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G,$$

with G_i normal in G_{i+1}, and such that each quotient group G_{i+1}/G_i is abelian.

a) Show that a subgroup of a solvable group is solvable.

b) Let D_n be the dihedral group of order $2n$. Show that D_n is solvable.

(5) Let $R = \{a + 3b\sqrt{-1} \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$.

a) Show that R is a domain.

b) Show that R is not a UFD. (*Hint*: Consider $(1 + 3\sqrt{-1})(1 - 3\sqrt{-1}) = 10$.)

(6) Determine with proof if the following polynomials are irreducible.

a) $x^4 + 3x^3 + x^2 - 2x + 1 \in \mathbb{Q}[x]$

b) $x^2y + xy^2 - x - y + 1 \in \mathbb{Q}[x, y]$

(7) For $k \in \mathbb{N}_{>0}$ set $\zeta_k = e^{\frac{2\pi i}{k}} \in \mathbb{C}$. Let $n, m \in \mathbb{N}_{>0}$ with $\mathrm{GCD}(n, m) = 1$.

a) Show $\mathbb{Q}(\zeta_n, \zeta_m) = \mathbb{Q}(\zeta_{nm})$.

b) Show $\mathrm{Gal}(\mathbb{Q}(\zeta_n, \zeta_m)/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$.

(8) Determine the splitting field of the following polynomial over the given field. Justify your answer.

a) $f(x) = x^6 + 3 \in \mathbb{F}_7[x]$.

b) $g(x) = x^5 + 2 \in \mathbb{F}_5[x]$.

(9) Let $\mathbb{K} \subseteq \mathbb{L}$ be a Galois extension and $G = \mathrm{Gal}(\mathbb{L}/\mathbb{K})$ with $|G| = 44$.

a) Determine the number of 11-Sylow subgroups of G.

b) Show there exists a surjective group homomorphism $\pi : G \to \mathbb{Z}/2\mathbb{Z}$.

c) Use (b) to show that there exists an intermediate field $\mathbb{K} \subseteq \mathbb{F} \subseteq \mathbb{L}$ such that $\mathbb{K} \subseteq \mathbb{F}$ is Galois and $[\mathbb{F} : \mathbb{K}] = 2$.