Algebra Prelim, June 8, 2022

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

(1) Let $A \in \mathbb{C}^{n \times n}$.

a) Prove that A and its transpose, A^{T}, have the same eigenvalues.

b) Provide an example showing that A and A^{T} do not necessarily have the same eigenvectors.

(2) Consider the group $G = \mathrm{GL}_4(\mathbb{C})$ and define the subset

$$\mathcal{M} = \{A \in G \mid A^2 = I_4\},$$

where I_4 is the identity matrix in G.

a) Let $A \in \mathcal{M}$ and $\lambda \in \mathbb{C}$ be an eigenvalue of A. Show that $\lambda \in \{1, -1\}$.

b) Show that every A in $\mathcal{M}$ satisfies $\mathrm{eig}(A; 1) \oplus \mathrm{eig}(A; -1) = \mathbb{C}^4$ and conclude that A is diagonalizable. (Here $\mathrm{eig}(A; \lambda)$ denotes the eigenspace of A associated to λ.)
[Hint: For $v \in \mathbb{C}^4$ the vector $v + Av$ will be useful.]

c) Consider the group action

$$G \times \mathcal{M} \longrightarrow \mathcal{M}, \ (S, A) \longmapsto S^{-1}AS.$$

Show that it partitions $\mathcal{M}$ into 5 orbits.

(3) Let G_1 and G_2 be finite groups of orders $n_1 > 1$ and $n_2 > 1$, respectively.

a) Suppose $\gcd(n_1, n_2) = 1$. Show that $\mathrm{Aut}(G_1 \times G_2) \cong \mathrm{Aut}(G_1) \times \mathrm{Aut}(G_2)$.
($\mathrm{Aut}(G)$ denotes the automorphism group of G).

b) Show that the statement in a) is not necessarily true if $\gcd(n_1, n_2) > 1$.

(4) Let G be a finite group and $\mathrm{Aut}(G)$ be its automorphism group. Recall the inner automorphism associated with $g \in G$ defined as

$$\phi_g : G \longrightarrow G, \ x \longmapsto gxg^{-1}.$$

Thus we have a well-defined group homomorphism $\phi : G \longrightarrow \mathrm{Aut}(G), \ g \longmapsto \phi_g$.

a) Characterize the groups G for which ϕ is injective.

b) Show that for $G = S_3$, the map ϕ is an isomorphism.
[Hint: Use that $S_3 = \langle (1\,2), (1\,2\,3) \rangle$ in order to show that $|\mathrm{Aut}(S_3)| \leq 6$.]

(5) Let R, S be commutative rings with identity and $\phi : R \longrightarrow S$ be a ring homomorphism such that $\phi(1) = 1$.

a) Let P be a prime ideal of S. Show that $\phi^{-1}(P)$ is a prime ideal of R.

b) Give an example showing that the pre-image of a maximal ideal need not be a maximal ideal.

(6) Let $R = \mathbb{Q}[x]/(x^{10} - 1)$.

a) Determine a direct product of fields that is isomorphic to R.

b) Determine the number of ideals of R.

(7) Consider the polynomials $f = x^2 - 2$ and $g = y^2 - 3$ with coefficients in $\mathbb{F}_5$. Note that both polynomials are irreducible. Give an explicit isomorphism from $\mathbb{F}_5[x]/(f)$ to $\mathbb{F}_5[y]/(g)$ and make sure to explain why it is well-defined.

(8) Let $E \,|\, F$ be a finite field extension and let R be a subring of E such that $F \subseteq R$. Show that R is a field.

(9) Let $E \,|\, F$ be a Galois extension of degree 55 with non-abelian Galois group $G := \mathrm{Gal}(E \,|\, F)$.

a) Determine the number of Sylow-p-subgroups of G for $p \in \{5, 11\}$.

b) Show that there exists exactly one intermediate field L with $E \neq L \neq F$ such that $L \,|\, F$ is Galois. Determine $[L : F]$.