Algebra Prelim, June 2, 2023

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even if you did not successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**
- **All problems carry the same weight, but the individual parts of a problem may have different weights.**

Good luck!

(1) Let V be an n-dimensional vector space over a field F and let $\phi : V \longrightarrow V$ be a linear map. Recall that the rank of ϕ, denoted by $\operatorname{rk}\phi$, is defined as the dimension of the image of ϕ. Suppose $\phi^2 = 0$. Show that $\operatorname{rk}\phi \leq n/2$.

(2) Let $A \in \mathbb{C}^{n\times n}$ be a unitary matrix, that is, $AA^* = I_n$, where $A^* = \bar{A}^{\mathsf{T}}$, that is, the complex conjugate transpose of A, and where I_n is the identity matrix in $\mathbb{C}^{n\times n}$.

a) Show that every eigenvalue of A has absolute value 1.

b) Show that if n is odd and all entries of A are in $\mathbb{R}$, then 1 or -1 is an eigenvalue of A.

(3) Let G be a finite group and K a normal subgroup of G. Furthermore, let P be a Sylow p-subgroup of K. Show the following.

a) For all $g \in G$, the group gPg^{-1} is conjugate to P in K.

b) $G = KN_G(P)$, where $N_G(P)$ is the normalizer of P in G.

(4) Let $\sigma = (1\ 2\ \ldots\ n) \in S_n$. Show that the centralizer of σ is the subgroup $\langle\sigma\rangle$.

(5) Consider the ring $R = \mathbb{Z}[\sqrt{-13}] = \{a + b\sqrt{-13} \mid a, b \in \mathbb{Z}\}$ and the norm function

$$N : \mathbb{R} \longrightarrow \mathbb{N}_0,\ a + b\sqrt{-13} \longmapsto a^2 + 13b^2.$$

You may use without proof that N is multiplicative.

a) Determine the group of units of R.

b) Show that $1 + \sqrt{-13}$ is irreducible.

c) Is R a UFD? Justify your answer.

(6) Show that a principal ideal in $\mathbb{Z}[x]$ is not maximal.
[Hint: Start with an ideal (f) and consider the cases $\deg f \leq 0$ and $\deg f \geq 1$ separately.]

(7) An element a in a commutative ring R is called a square if $a = b^2$ for some $b \in R$. Note that 0 is a square.

a) Let q be an odd prime power and $\mathbb{F}_q$ be a field with q elements. Show that the number of squares in $\mathbb{F}_q$ is $(q+1)/2$.
[Hint: Count the nonzero squares.]

b) Determine the number of squares in the ring $\mathbb{Z}/460\mathbb{Z}$.

(8) Let p be a prime number and F be a field with $\operatorname{char}(F) \neq p$. Suppose F contains a primitive p-th root of unity, denoted by ζ. Let $a \in F$ and $f = x^p - a$. Let E be a splitting field of f and θ be a root of f in E.

a) Show that all irreducible factors of f have the same degree.
[Hint: Any such factor is, up to a scalar, the minimal polynomial of some root of f.]

b) Conclude that f is either irreducible or the element a is contained in $F^p := \{c^p \mid c \in F\}$.

(9) Let $\zeta = e^{\frac{2\pi i}{12}} = \frac{1}{2}(\sqrt{3} + i)$ and consider $E = \mathbb{Q}(\zeta)$.

a) Show that $\sqrt{3} \in E$.

b) Present the Galois group of $E \mid \mathbb{Q}$ by describing each automorphism via its action on ζ.

c) Give the subgroup lattice of $\mathrm{Gal}(E \mid \mathbb{Q})$.

d) Give all subfields of E and present them in the form $\mathbb{Q}(\alpha)$ for some α.