ALGEBRA PRELIM, MAY 2024

- **Provide proofs of all statements, citing theorems that may be needed.**
- **Do as many problems as possible and present your solutions as carefully as possible.**
- **All problems carry the same weight, but the individual parts of a problem may have different weights.**

(1) Let A be an $n \times n$ matrix over an algebraically closed field. Recall that a matrix A is *nilpotent* if there exists a positive integer k such that $A^k = 0$.
   (a) Show that if A is nilpotent, then its only eigenvalue is 0.
   (b) Show that if the only eigenvalue of A is 0, then the characteristic polynomial of A is x^n.
   (c) Show that if the only eigenvalue of A is 0, then A is nilpotent.

(2) Let V be the vector space of polynomials of degree at most 4 with real coefficients in the variable x, and let $D\colon V \to V$ be the linear transformation given by $D(p) = \frac{dp}{dx}$.
   (a) Find the kernel of D^2.
   (b) Find the rank of D^2.

(3) Let

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_2 \right\}.$$

   (a) Show that H is a group under matrix multiplication.
   (b) Find the order of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (c) Show that the cyclic subgroup generated by A is normal in H.

(4) (a) Show that every group of order $88 = 2^3 \cdot 11$ has a normal subgroup of order 11.
   (b) Show that every group of order $2024 = 2^3 \cdot 11 \cdot 23$ has a normal subgroup of order $253 = 11 \cdot 23$.

(5) Let d be a squarefree integer, let

$$A = \begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix},$$

and let R be the subring of $M_2(\mathbb{Q})$ generated by A and the identity.

  (a) Compute the kernel of the homomorphism $\varphi\colon \mathbb{Q}[x] \to R$ given by $\varphi(f) = f(A)$.
  (b) Use part (a) to show that $\mathbb{Q}[\sqrt{d}]$ is isomorphic to R.

(6) Let F be a field and let $S = \{f(x) = a_0 + a_1 x + \cdots + a_n x^n \in F[x] \mid a_1 = 0\}$.

  (a) Show that S is a subring of $F[x]$.
  (b) Use the polynomial $x^2(x^2 - 1)$ to show that S is not a UFD.

(7) Let E be the splitting field of $x^6 + 1$ over $\mathbb{Q}$, and let ζ_{12} denote a primitive 12th root of unity.

  (a) Show that $E = \mathbb{Q}(\zeta_{12})$.
  (b) Identify $\mathrm{Gal}(E/\mathbb{Q})$.
  (c) Find all subfields of E.

(8) Let K be a field. Prove that, if $[K(\alpha) : K]$ is odd, then $K(\alpha) = K(\alpha^2)$.

(9) For each of the following, either provide an example or explain why it is not possible.

  (a) A field extension K of $\mathbb{Q}$ of degree 4 with $|\mathrm{Aut}(K/\mathbb{Q})| = 4$.
  (b) A field extension K of $\mathbb{Q}$ of degree 4 with $|\mathrm{Aut}(K/\mathbb{Q})| = 2$.
  (c) A field extension K of $\mathbb{Q}$ of degree 4 with $|\mathrm{Aut}(K/\mathbb{Q})| = 3$.