ALGEBRA PRELIM, JUNE 2025

- **Provide proofs of all statements, citing theorems that may be needed.**
- **Do as many problems as possible and present your solutions as carefully as possible.**
- **All problems carry the same weight, but the individual parts of a problem may have different weights.**

(1) Let V, W be vector spaces and let $T\colon V \to W$ be a function. The *graph* of T is the set

$$\{(v, T(v)) \in V \times W \mid v \in V\}.$$

Show that T is a linear transformation if and only if the graph of T is a subspace of $V \times W$.

(2) Let V be a finite-dimensional vector space over a field k, and let $T : V \to V$ be a linear map. Suppose that $\{v, Tv, T^2v, \cdots\}$ spans V for some $v \in V$. Prove that the minimal polynomial of T is equal to the characteristic polynomial of T.

(3) Let G be a group of order 380. Using the Sylow theorems, give the possible number of Sylow 5-subgroups and Sylow 19-subgroups. Prove that G must have either a normal subgroup of order 5 or a normal subgroup of order 19.

(4) $\mathrm{SL}(2, 3)$ is the group of 2×2 matrices over $\mathbb{F}_3$ with determinant 1.
   (a) What is the order of $\mathrm{SL}(2, 3)$?
   (b) Show that $\mathrm{SL}(2, 3)$ has a unique element of order 2.
   (c) Conclude that $\mathrm{SL}(2, 3)$ is not isomorphic to the symmetric group S_4.

(5) Let A be a commutative ring, and let $\varphi\colon A \to A$ be a surjective ring homomorphism. If A is Noetherian, show that φ is injective.

(6) Let k be a field and let $R = k[x]/(x^3)$.
  (a) Determine if $1 - x$ is a unit in R.
  (b) Find all zero divisors in R.

(7) Prove that $\mathbb{Q}(\sqrt[3]{2})$ is not a subfield of any cyclotomic extension of $\mathbb{Q}$.

(8) Let k be a field, and let $P(t), Q(t) \in k[t]$ be relatively prime polynomials. Prove that

$$\left[k(t)\colon k\left(\frac{P(t)}{Q(t)}\right)\right] = \max\{\deg(P), \deg(Q)\}.$$

(9) Let $f(x) = x^4 - 2 \in \mathbb{Q}[x]$ and let $K = \mathbb{Q}(2^{1/4}, i)$. Prove that K is a splitting field for $f(x)$.

Let σ be the element in G which maps $2^{1/4} \mapsto 2^{1/4}i$ and $i \mapsto i$. Let τ be the element in G which maps $2^{1/4} \mapsto 2^{1/4}$ and $i \mapsto -i$. It can be shown that

$$G = \{1, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\}.$$

For each of the following intermediate fields $\mathbb{Q} \subset E \subset K$, find the Galois group $\mathrm{Gal}(K/E)$ as a subgroup of G, and determine if E is Galois over $\mathbb{Q}$:
  (a) $\mathbb{Q}(2^{1/4})$
  (b) $\mathbb{Q}(i)$
  (c) $\mathbb{Q}(\sqrt{2}, i)$.