ANALYSIS PRELIMINARY EXAM

January 2, 2008

Instructions

1. This is a three hour examination which consists of two parts: Advances Calculus and Real *or* Complex Analysis. You should work problems from section on Advanced Calculus and from the section of the option that you have chosen.

2. You need to work a total of 5 problems, four mandatory problems (two mandatory problems from each part), and one optional problem from either advanced calculus or the option you have chosen. Please indicate clearly on your test paper whether you are taking the Real or Complex Analysis option, and which optional problems you are solving.

3. Do not put your name on any sheet except the cover page. The papers will be blind-graded.

4. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problems than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

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Advanced Calculus, Mandatory Problems

1. Let (X, d) be a metric space and $E \subset X$ be a subset.
(a) State the definition that a sequence of functions $f_n : E \to R$ uniformly converges to $f : E \to R$ on E.
(b) Prove that if $f_n : E \to R$ is continuous for $n = 1, 2, \cdots$, and f_n uniformly converges to f on E, then $f : E \to R$ is continuous.

2.
(a) Given a bounded, real-valued function $f : [0, 1] \to R$, state the definition that f is Riemann integrable on $[0, 1]$.
(b) Prove that if $f : [0, 1] \to R$ is monotonically increasing, then f is Riemann integrable on $[0, 1]$.

Advanced Calculus, Optional Problems

1. Suppose that $f : [0, 1] \to R$ is continuous and $f(1) = 0$. Prove that $\{x^n f(x)\}$ converges to zero uniformly on $[0, 1]$ as $n \to \infty$.

2. Let $E = [0, 1]$. Set

$$f(x) = \begin{cases} 0 & x \text{ is irrational} \\ 1 & x \text{ is rational.} \end{cases}$$

Since f is the characteristic function of the set Q of rational numbers in E and $m(Q) = 0$, the Lebesgue integral $\int_E f(x)\, dx = 0$. By considering upper and lower Riemann sums, show that the Riemann integral does not exist.

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Real Analysis, Mandatory Problems

1.
(a) State the definition of bounded variation for functions defined on $[0,1]$.
(b) Show that the function

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x > 0 \\ 0, & x = 0 \end{cases}$$

is of bounded variation on the interval $[0,1]$.

2.
(a) State Egoroff' s theorem.
(b) Prove that if $E \subset R$ is a measurable set of finite Lebesgue measure, $f_n : E \to R$ is a uniformly bounded sequence of measurable functions on E, and f_n converges to f point-wise on E, then

$$\lim_{n\to\infty} \int_E f_n(x)\, dx = \int_E f(x)\, dx.$$

Real Analysis, Optional Problems

1. Let f and f_k, $k = 1, 2, \cdots$ be measurable and finite a.e. on R^n. Suppose

$$\int_{R^n} |f_k - f|^2\, dx \to 0 \text{ as } k \to \infty$$

and

$$\int_{R^n} |f_k|^2\, dx \le M \text{ for all } k.$$

Show that

$$\int_{R^n} |f|^2\, dx \le M.$$

2. If $f(x)$ is summable on $E = [0,1]$, show that

$$\lim_{n\to\infty} \int_E f(x) \sin(nx)\, dx = 0.$$

(Hint: First prove the statement for step functions.)

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Complex Analysis, Mandatory Problems

1. Use the theory of residues to verify that

$$\int_0^\infty \frac{dx}{1+x^3} = \frac{2\pi}{3\sqrt{3}}.$$

(*Hint:* First integrate around the contour bounding the circular wedge

$$W = \left\{ z \in \mathbf{C} : |z| \leq R,\ 0 \leq \arg z \leq \frac{2\pi}{3} \right\},$$

and then let $R \to \infty$.)

2. Let f be an analytic function in $\Omega \setminus \{z_0\}$, with an isolated singularity at z_0.
(a) Explain how the behavior of $f(z)$ near z_0 determines the nature of the singularity.
(b) Show that e^f cannot have a pole at z_0.

Complex Analysis, Optional Problems

1. Suppose that f is an entire function such that for each $z_0 \in \mathbf{C}$ at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is equal to 0. Prove that f is a polynomial. (*Hint:* Note that $c_n n! = f^{(n)}(z_0)$ for each $n \geq 0$. Let $E_n = \left\{ z \in \mathbf{C} : f^{(n)}(z) = 0 \right\}$ and make use of a countability argument.)

2. Suppose that f is analytic in a deleted neighborhood of z_0 and that

$$|f(z)| \leq \frac{A}{|z - z_0|^{1-\varepsilon}}$$

for some $\varepsilon > 0$ and all z near z_0. Prove that f has a removable singularity at z_0.