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ANALYSIS PRELIMINARY EXAM

12 January 2009

NAME: ______________________________ OPTION: _______________

Instructions

1. This is a three hour examination which consists of two parts: **Advanced Calculus and Real** *or* **Complex Analysis**. You must work problems from the section on Advanced Calculus and from the section of the option, Reals or Complex, that you have chosen.

2. You need to work a total of 5 problems: Four mandatory problems (two mandatory problems from each of two parts), and one optional problem from either advanced calculus or the option you have chosen. Please indicate clearly on your test paper whether you are taking the **Real or Complex Analysis option**, and which optional problem you are solving. **Only 5 problems will be graded.**

3. Do not put your name on any sheet except the cover page. The papers will be blind-graded.

4. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to partial solutions of the easy parts of two different problems. Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. (a) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a metric space X with metric d_X. Let $\mathcal{L}$ be the set of all subsequential limit points of $\{x_n\}$. Prove that $\mathcal{L}$ is closed. If the metric space X is compact, prove that $\mathcal{L}$ is compact.

   (b) Let $X = \mathbf{R}$, with the Euclidean metric. Suppose that the sequence $\{x_n\}$ is bounded. Prove that $\mathcal{L}$ is nonempty.

2. (a) Define the Riemann integral of a real-valued, bounded function f on a bounded interval $[a, b] \subset \mathbf{R}$.

   (b) Let f and g be two continuous, real-valued functions on $[a, b]$. Suppose that $g \geq 0$. Prove that there exists a point $c \in [a, b]$ so that

$$\int_a^b fg = f(c) \int_a^b g.$$

Advanced Calculus, Optional Problems

1. Let (X, d_X) and (Y, d_Y) be two metric spaces and suppose that X is compact. If $f : X \to Y$ is continuous, prove that it is uniformly continuous.

2. Prove the following: A sequence of real-valued functions f_k is uniformly convergent on a metric space (X, d_X) if and only if for all $\epsilon > 0$, there exists a constant $N > 0$ so that if $n, m > N$, one has

$$|f_n(x) - f_m(x)| < \epsilon,$$

for all $x \in X$.

Real Analysis, Mandatory Problems

1. (a) State the Lebesgue Dominated Convergence Theorem (LDCT) for functions on $\mathbf{R}^n$.

   (b) Construct a sequence of nonnegative, Lebesgue integrable functions $f_k : [0,1] \to [0,\infty)$ such that
   $$\int_0^1 f_k = 1, \quad \text{and} \quad f_k \to 0 \ a.e. \text{ in } [0,1].$$

   (c) Apply the LDCT to prove that if $f \in L^1(\mathbf{R}^n)$, then
   $$\lim_{k\to\infty} \int_{\mathbf{R}^n} f(x) e^{-\frac{|x|^2}{k}}\, dx = \int_{\mathbf{R}^n} f(x)\, dx.$$

2. (a) Let $E \subset \mathbf{R}^n$ be a Lebesgue measurable set. Let $\{f_k\}$ and f be Lebesgue measurable functions on E that are finite a.e. Give the definition of what it means for the sequence $\{f_k\}$ to *converge in measure* to f on E.

   (b) Prove that if the Lebesgue measure of E is finite, that is $|E| < \infty$, and $f_k \to f$ a.e. on E, then the sequence $\{f_k\}$ converges to f in measure on E.

Real Analysis, Optional Problems

1. Suppose that $f : \mathbf{R} \to [0,\infty)$ is a nonnegative, Lebesgue measurable function and that $\int_{\mathbf{R}} f = 1$. For $a > 1$, define
   $$G_a(x) \equiv \sum_{n=1}^{\infty} f(a^n x), \ x \in \mathbf{R}.$$
   Apply the Monotone Convergence Theorem to evaluate the integral:
   $$\int_{\mathbf{R}} G_a(x)\, dx.$$

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2. For a nonnegative Lebesgue measurable function $f : [0,1] \to [0,\infty)$, define the function g by

$$g(t) \equiv |\{x \in [0,1] : f(x) > t\}|,$$

where $|E|$ denotes the Lebesgue measure of the set $E \subset \mathbf{R}$. Suppose that, in addition, $f \in L^p([0,1])$, for some $1 < p < \infty$. Apply Tchebyshev's inequality to prove that

$$\int_0^\infty g(t)\, dt < \infty.$$

Hint: Tchebyshev's inequality: Suppose $f \geq 0$ and integrable. If $\alpha > 0$ and $E_\alpha = \{x : f(x) > \alpha\}$, then

$$|E_\alpha| \leq \frac{1}{\alpha} \int f.$$

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Complex Analysis, Mandatory Problems

1. Let $f(z) = (1 + z^n)^{-1}$, for some $n \geq 2$ integer. Let γ denote the path consisting of the segment $0 \leq x \leq R$ on the real axis, the arc $z = Re^{it}$, with $0 \leq t \leq (2\pi)/n$ on the circle $|z| = R$, and the segment from $Re^{(2\pi i)/n}$ to 0 on the line $\arg z = (2\pi)/n$.

   (a) Compute
   $$\int_\gamma f(z)\, dz.$$

   (b) Let $R \to \infty$ and find
   $$\int_0^\infty \frac{1}{1+x^n}\, dx.$$

2. Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be a power series with a nonzero radius of convergence. Suppose that $a_1 \neq 0$.

   (a) Prove that $|a_1| > \sum_{n=2}^\infty n|a_n|r^{n-1}$, for some $r > 0$ sufficiently small.

   (b) Prove that f is one-to-one on the disk $|z| < r$. *Hint:* Consider the expression $f(z_1) - f(z_2)$ as an infinite series and show that it is nonzero for any pair of distinct points $z_1 \neq z_2$ with $|z_1| < r$ and $|z_2| < r$.

   (c) Use the results of parts (a) and (b) to prove that if f is analytic on some domain $\Omega \subset \mathbf{C}$ with $f'(z_0) \neq 0$, for some $z_0 \in \Omega$, then f is one-to-one in some neighborhood of z_0.

Complex Analysis, Optional Problems

1. Let $f(z)$ be analytic in a deleted neighborhood $\mathcal{N}$ of a point ζ_0, where $\mathcal{N} = \{z \in \mathbf{C} \mid 0 < |z - \zeta_0| < r\}$, for some $r > 0$. Suppose that $\Re f > 0$ (the real part of f) on $\mathcal{N}$. Prove that f has a removable singularity at ζ_0.

2. Let f be analytic on the open unit disk D and continuous on its closure $\overline{D}$. If $|f(z)| = 1$ on the boundary of the disk ∂D, prove that f is a rational function. *Hint:* First consider the case when $f \neq 0$ in D.