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ANALYSIS PRELIMINARY EXAM

8 January 2010

NAME: ______________________________ OPTION: _______________

Instructions

1. This is a three hour examination which consists of two parts: **Advanced Calculus and Real** *or* **Complex Analysis**. You must work problems from the section on Advanced Calculus and from the section of the option, Reals or Complex, that you have chosen.

2. You need to work a total of 5 problems: Four mandatory problems (two mandatory problems from each of two parts), and one optional problem from either advanced calculus or the option you have chosen. Please indicate clearly on your test paper whether you are taking the **Real or Complex Analysis option**, and which optional problem you are solving. **Only 5 problems will be graded.**

3. Do not put your name on any sheet except the cover page. The papers will be blind-graded.

4. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to partial solutions of the easy parts of two different problems. Indicate clearly what theorems and definitions you are using.

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Advanced Calculus, Mandatory Problems

1. Let (X, d_X) be a metric space, D a dense subset of X, and (Y, d_Y) a complete metric space. Prove that every uniformly continuous function $f : D \to Y$ has a unique extension $\overline{f} : X \to Y$ which is uniformly continuous and satisfies $\overline{f}(x) = f(x)$ for each $x \in D$.

2. (a) Let f be a bounded real-valued function on the interval $[a, b]$. Give the definition of $\int_a^b f\,dx$, the Riemann integral of f on $[a, b]$.

   (b) Suppose f is defined on $[0, 1]$ by

   $$\begin{aligned} f(1/n) &= 1 \qquad n \in N \\ f(x) &= 0 \qquad \text{otherwise} \end{aligned}$$

   Prove (without using Lebesgue theory) that f is Riemann integrable on $[0, 1]$ and compute the integral.

Advanced Calculus, Optional Problems

1. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is said to be a Lipschitz function (with constant M) if there is a constant $M > 0$ such that

   $$d_Y(f(x_1), f(x_2)) \leq M d_X(x_1, x_2) \qquad \text{for all} \quad x_1, x_2 \in X.$$

   For each $n = 1, 2, 3, \cdots$, let $g_n : X \to Y$ be a Lipschitz function with constant M_n.

   (a) Suppose $\sup M_n < \infty$ and $\{g_n\}$ converges uniformly to $g : X \to Y$ on X. Prove that g is a Lipschitz function.

   (b) Show that $f(x) = \sqrt{x}$ on $[0, 1]$ is not a Lipschitz function. Using this function prove that there is a sequence of Lipschitz functions $\{f_k\}$ on $[0, 1]$ converging uniformly to f on $[0, 1]$ but for which the Lipschitz constants $M_n \to \infty$.

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2. Every rational number x can be written in the form $x = m/n$, where $n > 0$, and m and n are integers without any common divisor. When $x = 0$, we take $n = 1$. Consider the function f defined on $[0,1]$ by

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational,} \\ m \sin \frac{1}{n} & \text{if } x = \frac{m}{n}. \end{cases}$$

Determine the set of points in $[0,1]$ on which f is continuous. Justify your answer.

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Real Analysis, Mandatory Problems

1. (a) Define a Lebesgue measurable subset of R^n.

   (b) Prove that a subset $E \subset R^n$ is Lebesgue measurable if and only if there is a G_δ set $F \subset \mathbb{R}^n$ so that E and F differ by a set of measure zero.

2. (a) Let f be a bounded, real-valued, measurable function on a measurable set $A \subset R^n$. Prove that for any $\epsilon > 0$, there exist simple functions f_j, for $j = 1, 2$ so that $f_1 \leq f \leq f_2$ and $f_2 - f_1 \leq \epsilon$.

   (b) Suppose that f_k is an increasing sequence of measurable functions on a measurable set $E \subset \mathbb{R}^n$. Suppose that $f_1 \in L^1(E)$ and that $f_k \to f$ almost everywhere in E. Prove that $f \in L^1(E)$ if and only if $\lim_k \int_E f_k < \infty$.

Real Analysis, Optional Problems

1. Suppose that $\{E_k\}_{k=1}^{\infty}$ is a countable family of Lebesgue measurable subsets of R^n and that the Lebesgue measures of these sets are summable:

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

   Define a set $E \subset R^n$ by

$$E = \{x \in R^n \mid x \in E_k \text{ for infinitely many } k\}.$$

   Prove that E is measurable and that $m(E) = 0$.

2. Let $b_k(x) \geq 0$ be a countable family of nonnegative measurable functions on $\mathbb{R}^n$.

   (a) Prove that

$$\int_{\mathbb{R}^n} \sum_{k=1}^{\infty} b_k(x)\, dx = \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} b_k(x)\, dx.$$

   (b) If

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} b_k(x)\, dx < \infty,$$

   prove that the series $\sum_{k=1}^{\infty} b_k(x)$ converges almost everywhere.

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Complex Analysis, Mandatory Problems

1. Let D_1 and D_2 be arbitrary domains in the complex plane and let $f : D_1 \to D_2$ be analytic in D_1. Prove that if u is harmonic in D_2 then the composition $u \circ f$ is harmonic in D_1.

2. Compute $\displaystyle\int_0^\infty \frac{x}{x^n+1}\,dx$ for $n > 2$ by integrating a function around the boundary of a circular sector with central angle of $\frac{4\pi}{n}$ at the origin, radius R, and a side on the real axis. Prove that the line integral over the circular part of the boundary of the sector approaches 0 as $R \to \infty$.

Complex Analysis, Optional Problems

1. Show that if $p(z)$ is a polynomial of degree at least two, then the sum of the residues of $f(z) = \dfrac{1}{p(z)}$ at each of the zeros of $p(z)$ is zero.

2. Let $D = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Show that there does not exist an analytic function $F(z)$ in D such that $F'(z) = \frac{1}{z}$ for all z in D.