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Analysis Preliminary Exam

4 January 2012

Instructions

1. This is a three hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis. You must work problems from the Advanced Calculus section and the Real Analysis section or the Complex Analysis section, depending on the option you have chosen.
2. You should attempt a total of five problems: four mandatory problems (two from each section) and one optional problem from either section. Please indicate clearly whether you are taking the Real Analysis option or the Complex Analysis option and which optional problem is to be graded. If you do not indicate which optional problem is to be graded, the one with the lowest score will be used to determine your grade.
3. Do not put your name on any sheet except the cover sheet. The exam will be blind-graded.
4. Each question is weighted equally.
5. You should provide complete and detailed solutions to each problem that you work. More weight will be given for a complete solution to one problem than for solutions of the easy bits in two different problems.
6. Indicate clearly the definitions and theorems that you are using.

Advanced Calculus, Mandatory Problems

1. Let f be a continuous real-valued function on $[0, \infty)$. Suppose that $\lim_{x \to \infty} f(x)$ exists and is finite. Show that f is uniformly continuous on $[0, \infty)$.

2. Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Suppose that $a < b$. Prove that

$$\lim_{p \to \infty} \left(\int_a^b |f(x)|^p \, dx \right)^{1/p} = \max_{x \in [a,b]} |f(x)|.$$

Advanced Calculus, Optional Problems

3. Let $\{a_n\}$ be a sequence of positive numbers. Show that

$$\liminf_{n \to \infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n \to \infty} \sqrt[n]{a_n}.$$

4. Let f be defined and differentiable on an interval $[a, b]$ in such a way that $f(a) = 0$ and

$$|f'(x)| \leq A|f(x)|$$

for all $x \in [a, b]$ and some constant A. Prove that $f \equiv 0$ on $[a, b]$.

Real Analysis

Mandatory problems

If E is a Lebesgue measurable set, we let $m(E)$ denote the Lebesgue measure of E.

1. (a) Say what it means for a set $E \subset \mathbf{R}$ to be measurable.

   (b) Show that a countable union of measurable sets is measurable.

2. (a) Construct a sequence of real-valued measurable functions $\{f_k\}_{k=1}^{\infty}$ on the real line $\mathbf{R}$ so that $\lim_{k\to\infty} f_k(x) = 0$ for all x, but

$$\lim_{k\to\infty} \int_{\mathbf{R}} f_k(x)\,dx = 1.$$

   (b) Suppose that f is Lebesgue integrable on the real line. Prove that

$$\lim_{\alpha\to 0^+} \int_{\mathbf{R}} f(x)e^{-\alpha x^2}\,dx = \int_{\mathbf{R}} f(x)\,dx.$$

Optional problems

3. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is continuous and $\Gamma = \{(x, f(x)) : x \in \mathbf{R}\}$ is the graph of f. Show that Γ is a set of measure zero in $\mathbf{R}^2$.

4. Let $\{E_k\}_{k=1}^{\infty}$ be a family of measurable subsets of the real line and suppose that $\sum_{k=1}^{\infty} m(E_k) < \infty$. Let E denote the collection of points which lie in infinitely many of the sets E_k. Show that $m(E) = 0$.

Complex Analysis

Mandatory problems

1. Let r be fixed with $0 < r < 1$. Use the residue theorem to show that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos\theta + r^2}\, d\theta = 1.$$

2. Determine the number of zeroes of the polynomial

$$z^7 + 4z^4 + z^3 + 1$$

lying in the disk $|z| < 1$, and in the annulus $1 < |z| < 2$.

Optional problems

3. Let $f : \mathbf{C} \to \mathbf{C}$ be a one-to-one analytic map of the entire complex plan $\mathbf{C}$ onto itself. Prove that f is linear; that is, $f(z) = az + b$ for appropriate constants a and b. Hint: Consider the Laurent expansion

$$f(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \dots$$

for the function f about the point at ∞.

4. Let $f(z)$ be defined by the power series

$$f(z) = \sum_{n=0}^{\infty} z^{n!}, \qquad \text{for } |z| < 1.$$

(a) Prove that the series has radius of convergence equal to 1.

(b) Prove that $f(z)$ cannot be continued analytically to a neighborhood of any point on the circle of convergence $|z| = 1$. Hint: Prove that

$$\lim_{r\to 1^-} |f(re^{i\theta})| = +\infty$$

for θ in a dense subset of $[0, 2\pi]$.