Preliminary Examination in Analysis

January 2, 2013

Instructions

- This is a three-hour examination which consists of two parts: Advanced Calculus and Real Analysis.
- You are to work a total of five problems (four mandatory problems and one optional problem).
- Please indicate clearly on your test paper which optional problem is to be graded.
- Indicate clearly what theorems and definitions you are using.

On 12/p2

Advanced calculus
Mandatory problems

1. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers and suppose that

$$\limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1.$$

Show that the series $\sum_{n=0}^{\infty} a_n$ converges to a finite value.

2. Let (X, d) be a metric space and let E be a subset of X. Define

$$\rho_E(x) = \inf\{d(x,y) : y \in E\}.$$

(a) Show that $\rho_E(x) = 0$ if and only if x belongs to $\bar{E}$, the closure of E

(b) Show that ρ_E is uniformly continuous by showing that

$$|\rho_E(x) - \rho_E(y)| \le d(x,y)$$

(be sure to explain why this inequality implies uniform continuity).

Optional problems

3. Let f be the function which is defined by $f(p/q) = 1/q$ if p and q are non-zero integers with no common factor and $f(x) = 0$ if x is irrational or $x = 0$.

Is f Riemann integrable on $[0, 1]$?

4. This problem concerns real-valued continuous functions.

(a) State the Weierstrass Approximation Theorem.

(b) Suppose that f is a real-valued continuous function on $[0, 1]$ with the property that $\int_0^1 x^n f(x)\, dx = 0$ for all $n = 0, 1, 2, \ldots$. Prove that f is the zero function. You may assume the following theorem: If g is continuous and nonnegative on $[a, b]$ and $\int_a^b g(x)\, dx = 0$, then g is the zero function.

Sm12/p3

REAL ANALYSIS
MANDATORY PROBLEMS

If E is a Lebesgue measurable set, we let $m(E)$ denote the Lebesgue measure of E.

1. (a) Give the definition of the exterior measure $m_*(E)$ for $E \subset \mathbf{R}$.
   (b) Show that the exterior measure is countably sub-additive.

2. Do we have

$$\sum_{n=0}^{\infty} \int_{\mathbf{R}} \frac{x^n}{n!} \exp(-x^2)\, dx = \int_{\mathbf{R}} \exp(-x^2 + x)\, dx?$$

OPTIONAL PROBLEMS

3. Can you find a Lebesgue measurable set $E \subset \mathbf{R}$ so that for each interval I, $m(E \cap I) = \frac{1}{2} m(I)$?

4. We say that $E \subset \mathbf{R}$ is of content 0 if for each $\epsilon > 0$, we may find a covering of E by a finite collection of closed intervals $\{[a_j, b_j] : j = 1, \ldots, N\}$ so that $E \subset \cup_{j=1}^{N} [a_j, b_j]$ and $\sum_{j=1}^{N} b_j - a_j < \epsilon$.
   (a) If a set is of content zero, must it be of measure zero?
   (b) If a set is of measure zero, must it be of content zero?

Jan 12/04

Complex Analysis
Mandatory problems

1. Show that there is no function f analytic in the open unit disk $\{z : |z| < 1\}$ such that
$$f(1/n) = \frac{(-1)^n}{n^2}, \qquad \text{for } n = 2, 3, 4, \ldots$$

2. Suppose Ω is a bounded plane domain and that
$$f : \bar{\Omega} \to \mathrm{C}$$
is a non-constant function analytic in Ω and continuous in $\bar{\Omega}$ such that $|f(z)| = 1$ for all $z \in \partial\Omega$. Prove that $f(z_0) = 0$ for some $z_0 \in \Omega$.

Optional problems

3. Let f be an entire function with the property that
$$|f(z)| \geq |z|^N, \qquad \text{when } |z| > M$$
for some positive integer N and some $M > 0$. Prove that f is a polynomial.

4. Given a real number $a > 1$, show that
$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta}\, d\theta = \frac{2\pi}{\sqrt{a^2 - 1}}.$$