Preliminary Examination in Analysis

January 2016

Instructions

- This is a three-hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis.
- You should work problems from the section on advanced calculus and from the section of the option you have chosen.
- You are to work a total of five problems (four mandatory problems and one optional problem).
- You must work two mandatory problems from each part.
- Please indicate clearly on your test paper which optional problem is to be graded.
- Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. Let $\{a_n\}$ and $\{b_n\}$ be two bounded sequences of real numbers. Show that if $\{a_n\}$ is convergent, then

$$\limsup_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

2. Let $\{f_n\}$ be a sequence of real-valued continuous functions on a metric space X. Suppose that f_n converges uniformly to f on X. Show that f is continuous on X.

Advanced Calculus, Optional Problems

3. Let f be a real-valued continuous function on $\mathbb{R}$. Suppose that

$$f(x+y) + f(x-y) = 2\big[f(x) + f(y)\big]$$

for any $x, y \in \mathbb{R}$. Show that there exists α such that $f(x) = \alpha x^2$ for all $x \in \mathbb{R}$.

4. Let $\{a_n\}$ be a decreasing sequence of nonnegative real numbers. Suppose that $\lim_{n\to\infty} a_n = 0$. Also assume that the partial sum sequence $\{B_n\}$ of the series $\sum_{n=1}^{\infty} b_n$ is bounded. Show that the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

Real Analysis, Mandatory Problems

1. (a) State the definition of Lebesgue exterior measure $m_*(E)$ of a subset E of $\mathbb{R}^d$.
(b) Use the definition in part (a) to show that if E_1 and E_2 are two subsets of $\mathbb{R}^d$ and $\text{dist}(E_1, E_2) > 0$, then

$$m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2).$$

Note that E_1 and E_2 are not assumed to be measurable.

2. Let $f = f(x, y)$ be a function on $\{(x, y) : 0 \leq x, y \leq 1\}$. Suppose that for each x, $f(x, y)$ is an integrable function of y, and that $\partial f(x, y)/\partial x$ is a bounded function of (x, y). Show that $\partial f(x, y)/\partial x$ is a measurable function of y for each x and

$$\frac{d}{dx} \int_0^1 f(x, y)\, dy = \int_0^1 \frac{\partial}{\partial x} f(x, y)\, dy.$$

Real Analysis, Optional Problems

3. Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set $E \subset \mathbb{R}^d$ with $m(E) < \infty$. Suppose that for each $x \in E$,

$$\sup \left\{ |f_k(x)| : k \geq 1 \right\} = M_x < \infty.$$

Show that for each $\varepsilon > 0$, there exists a closed set F such that $m(E \setminus F) < \varepsilon$ and

$$\sup \left\{ |f_k(x)| : x \in F \text{ and } k \geq 1 \right\} = M < \infty.$$

4. Let $f(x, y)$ be a nonnegative measurable function in $\mathbb{R}^2$. Suppose that for a.e. $x \in \mathbb{R}$, $f(x, y)$ is finite for a.e. $y \in \mathbb{R}$. Show that for a.e. $y \in \mathbb{R}$, $f(x, y)$ is finite for a.e. $x \in \mathbb{R}$.

Complex Analysis, Mandatory Problems

In the following problems, $B(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$.

1. Briefly outline the proof of the Cauchy-Goursat Theorem: If f is analytic in an open set containing the rectangle,

$$R = \{z = x + iy : a \le x \le b \text{ and } c \le y \le d\},$$

then

$$\int_{\partial R} f(z)\, dz = 0. \tag{1}$$

In the proof you may only assume the definition of analyticity, basic properties of line integrals, and the fact that (1) is valid when $f(z) = Az + B$ and A, B are constants.

2. Find

$$\int_{\partial B(0,1)} \frac{|dz|}{|z-a|^2},$$

where $\partial B(0,1)$ is oriented counterclockwise and a is a complex number with $|a| \neq 1$.

Complex Analysis, Optional Problems

3. Find a Möbuis transformation T mapping $B(0,1) \setminus \overline{B(3/4, 1/4)}$ onto

$$\{z = x + iy : 0 < x < 1\}.$$

4. Prove the following generalization of Liouville's Theorem: If f is an entire function and $|f(z)| \le 1 + |z|^{1/2}$ for all $z \in \mathbb{C}$, then f is constant.