Preliminary Examination in Analysis

January 2017

Instructions

- This is a three-hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis.
- You should work problems from the section on advanced calculus and from the section of the option you have chosen.
- You are to work a total of five problems (four mandatory problems and one optional problem).
- You must work two mandatory problems from each part.
- Please indicate clearly on your test paper which optional problem is to be graded.
- Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and f' is continuous. Show that the restriction of f to any closed interval $[a, b]$ is Lipschitz continuous.

2. Suppose that (X, d) is a metric space, fix a point a, and let $f(x) = d(a, x)$. Show that the function $f : X \to \mathbb{R}$ is uniformly continuous.

Advanced Calculus, Optional Problems

3. Let $f : (a, b) \to \mathbb{R}$ be differentiable. Suppose that $f'(c) = 0$ for some $c \in (a, b)$. Show that if f has a local minimum at $x = c$ and $f''(c)$ exists, then $f''(c) \geq 0$.

4. Suppose that $f \in C[0, 1]$ and

$$\int_a^b x^n f(x)\, dx = 0 \quad \text{for all integer } n \geq 0.$$

Show that f is the zero function.

Real Analysis, Mandatory Problems

1. For $f \in L^1(\mathbb{R})$, the Fourier transform of f is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi}\, dx.$$

   (1) Show that $\widehat{f}$ is continuous in $\mathbb{R}$.

   (2) Show that $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$. For this part, you may assume that the set of all linear combinations of characteristic functions over bounded open intervals is dense in $L^1(\mathbb{R})$.

2. Let f be an integrable function in $\mathbb{R}^d$. Show that

$$\lim_{\alpha \to \infty} \alpha\, m\big\{x \in \mathbb{R}^d : |f(x)| > \alpha\big\} = 0.$$

Real Analysis, Optional Problems

3.
   (1) State Egorov's Theorem.

   (2) Use Egorov's Theorem to prove the Bounded Convergence Theorem: if $\{f_k\}$ is a sequence of measurable functions on a measurable set E with $m(E) < \infty$, such that $f_k \to f$ a.e. in E and $|f_k| \leq M$ a.e. in E for some finite constant M, then

$$\int_E |f_k - f|\, dx \to 0 \quad \text{as } k \to \infty.$$

4. For a measurable function f on a measurable set $E \subset \mathbb{R}$, define

$$\|f\|_{L^\infty(E)} = \inf\Big\{\alpha : m\{x \in E : |f(x)| > \alpha\} = 0\Big\}.$$

Show that if $\|f\|_{L^\infty(E)} < \infty$ and $0 < m(E) < \infty$, then

$$\lim_{p \to \infty} \left(\frac{1}{m(E)} \int_E |f|^p\, dx\right)^{1/p} = \|f\|_{L^\infty(E)}.$$

Complex Analysis, Mandatory Problems

1. Use the residue theorem to verify that

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi}{n\sin(\pi/n)},$$

whenever $n = 2, 3, 4, \ldots$.

2. Let f be a non-constant entire function. Show that the range of f is dense in $\mathbb{C}$.

Complex Analysis, Optional Problems

3. Suppose that f has a power series expansion about 0 which converges in all of $\mathbb{C}$, and that

$$\iint_{\mathbb{C}} |f(x+iy)|\, dxdy < \infty.$$

Prove that $f \equiv 0$.

4. Suppose that f is analytic in $|z| < 1$ and continuous on $|z| \leq 1$. Prove that if $f \equiv 0$ on some bounded arc I, no matter how small the arc, then $f \equiv 0$ on the entire disk $|z| \leq 1$.