Preliminary Examination in Analysis

January 2018

Instructions

• This is a three-hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis.

• You should work problems from the section on advanced calculus and from the section of the option you have chosen.

• You are to work a total of five problems (four mandatory problems and one optional problem).

• You must work two mandatory problems from each part.

• Please indicate clearly on your test paper which optional problem is to be graded.

• Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. Let $L \in \mathbb{R}$. A sequence $\{a_n\}$ is said to have Cesaro sum L if the partial sum sequence $\{s_n\}$ satisfies the relation

$$\lim_{n\to\infty} \frac{s_1 + s_2 + \cdots + s_n}{n} = L.$$

Show that if

$$\sum_{n=1}^{\infty} a_n = L,$$

then $\{a_n\}$ has Cesaro sum L.

2. Suppose $\{f_n\}$ is a sequence of continuous functions on $[0,1]$ so that $f_n \to f$ pointwise on $[0,1]$, where f is also continuous on $[0,1]$. Is it true that

$$\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx?$$

If the answer is yes, then prove it; if no, give a counterexample.

Advanced Calculus, Optional Problems

3. Let $\{f_n\}$ be a sequence of functions defined on $E \subset \mathbb{R}$ with values in $\mathbb{R}$. Show that $\{f_n\}$ converges uniformly on E if and only if for every $\varepsilon > 0$, there exists an integer N such that

$$|f_n(x) - f_m(x)| < \varepsilon,$$

whenever $m \geq N$, $n \geq N$, and $x \in E$.

4. Let f be a continuous real-valued function defined on a closed interval $[a,b]$. Suppose that $a < b$. Prove that

$$\lim_{p\to\infty} \left(\int_a^b |f(x)|^p\,dx \right)^{1/p} = \max_{x\in[a,b]} |f(x)|.$$

Real Analysis, Mandatory Problems

1. Let $A = [0,1] \times [0,1]$ and $f_n : A \to \mathbb{R}$ be a sequence of measurable, uniformly bounded functions so that, for a.e. $x \in [0,1]$, $f_n(x,y) \to xy$, pointwise in y. Let $d\mu$ denote the Lebesgue measure on $\mathbb{R}^2$. Show that

$$\lim_{n\to\infty} \int_A f_n(x,y)\, d\mu$$

exists, and compute the limit. Be sure to justify your steps.

2. Suppose $A \subset \mathbb{R}$ has positive outer measure. Let $0 < \alpha < 1$. Show that there exists an interval I such that

$$m^*(A \cap I) \geq \alpha\, m^*(I).$$

Real Analysis, Optional Problems

3. Suppose $f, g : [0,1] \to \mathbb{R}$ are absolutely continuous functions. Show that their product fg is also absolutely continuous on $[0,1]$.

4. This problem concerns measurable functions $f : E \subset \mathbb{R} \to \mathbb{R}$.

(a) State Egoroff's Theorem.

(b) Suppose that $E \subset \mathbb{R}$ is a measurable set with $m(E) < \infty$, and that $\{f_n\}_{n=1}^{\infty}$ is a sequence of uniformly bounded, real-valued, measurable functions on E with $f_n \to f$ pointwise for a.e. x. Using Egoroff's Theorem, prove that

$$\lim_{n\to\infty} \int_E f_n(x)\, dx = \int_E f(x)\, dx.$$