Preliminary Examination in Analysis

January 2019

Instructions

This is a three-hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis.

You should work on problems from the section on advanced calculus and from the section of the option you have chosen.

You are to work a total of **five problems**: *four* mandatory problems and *one* optional problem. You must work *two* mandatory problems from each part.

Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

Problem 1. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences, and $\lim_{n\to\infty} a_n b_n = 0$. Show that at least one of $\{a_n\}$ or $\{b_n\}$ has a subsequence that converges to zero.

Problem 2. Let $\{a_n\}$ be a sequence which converges to 0. Suppose $f : [0,1] \to \mathbb{R}$ is bounded, and f is continuous except on the set $\{x \in [0,1] | x = a_n \text{ for some } n\}$. Show that f is Riemann integrable on $[0,1]$. You should only use the definition and properties of the Riemann integral from advanced calculus, and you may NOT use any theorems from the real analysis class.

Advanced Calculus, Optional Problems

Problem 3. Let f be positive and continuous on $[0,\infty)$ and differentiable on $(0,\infty)$. Suppose $|f'(x)| < 1$ for all $x \in (0,\infty)$ and $\int_0^\infty f(x)\,dx$ is finite. Show that f is bounded on $[0,\infty)$.

Problem 4. Suppose $\{a_n\}$ is a positive sequence, and $\sum_{n=1}^{\infty} a_n$ converges. Show that if g is continuously differentiable and $g(0) = 0$, then $\sum_{n=1}^{\infty} g(a_n)$ converges.

Real Analysis, Mandatory Problems

Problem 1. Let $f \in L^1(\mathbb{R})$, and define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = e^{-nx^2} f(x), \quad n \geq 1$$

Show that $f_n \in L^1(\mathbb{R})$, and find $\lim_{n\to\infty} \int_{\mathbb{R}} f_n$.

Problem 2. Show that if $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous and Lebesgue integrable, then

$$\lim_{|x|\to\infty} f(x) = 0$$

Real Analysis, Optional Problems

Problem 3. Show that an absolutely continuous function on a bounded interval $[a, b]$ is also of bounded variation on $[a, b]$.

Hint: For an arbitrary partition, look at a refinement of it with a fixed partition of norm at most δ for a suitably chosen δ.

Problem 4. Suppose $E \subset \mathbb{R}$ is measurable, and there is $f \in L^1(E)$ so that $f > 0$ almost everywhere, and $\int_E f = 0$. Show that $m(E) = 0$.

Complex Analysis, Mandatory Problems

Problem 1. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function such that $|f(z)| \geq |z|$ for all $z \in \mathbb{C}$. Prove that $f(z) = cz$ for some constant $c \in \mathbb{C}$ with $|c| \geq 1$.

Problem 2. Use a semicircle contour and the residue theorem to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1} dx.$$

Complex Analysis, Optional Problems

Problem 3. Find the number of roots of the equation $z^4 + 6z + 3 = 0$ in the annulus $1 < |z| < 2$.

Problem 4.

(a) Does there exist a non-constant entire function $f\colon \mathbb{C} \to \mathbb{C}$ such that $f(n) = 0$ for all positive integers n?

(b) Does there exist a non-constant entire function $f\colon \mathbb{C} \to \mathbb{C}$ such that $f(1/n) = 0$ for all positive integers n?

Make sure to justify your answers!