Preliminary Examination in Analysis

January 6, 2020

Instructions

This is a three-hour examination which consists of two parts:

(1) Advanced Calculus

(2) Real or Complex Analysis.

You should work on problems from the section on advanced calculus and from the section of the option you have chosen.

You are to work a total of **five problems**: *four* mandatory problems and *one* optional problem. You must work *two* mandatory problems from each part.

Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy parts from two different problems. Be sure to indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

Problem 1. Suppose that $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 0} g(x) = L$ for some $L \in \mathbb{R}$. Show that $\lim_{x\to 0}(f(x) + g(x)) = \infty$.

Problem 2. Suppose that f is continuous on $[0, \infty)$ and that $\lim_{x\to\infty} f(x) = 0$. Show that f is uniformly continuous on $[0, \infty)$.

Advanced Calculus, Optional Problems

Problem 3. Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and nondecreasing, i.e., $f(x_1) \leq f(x_2)$ whenever $x_1 \leq x_2$ and $|f(x)| \leq M$ for all $x \in [a, b]$ and some $M > 0$.

(a) Show that the limits
$$f(x^+) = \lim_{y\to x,\, y>x} f(y)$$
exist for every $x \in [a, b)$ and that the limits
$$f(x^-) = \lim_{y\to x,\, y<x} f(y)$$
exist for every $x \in (a, b]$.

(b) Show that f has at most countably many jump discontinuities. *Hint*: this is equivalent to showing that the range of f is an interval with at most countably many subintervals removed.

Problem 4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and $f(0) = f'(0) = 0$. Show that if $\{a_n\}$ is nonnegative and
$$\sum_{n=1}^{\infty} a_n \text{ converges}$$
then
$$\sum_{n=1}^{\infty} f(a_n) \text{ converges.}$$

Real Analysis, Mandatory Problems

Problem 1. Suppose that f is a finite-valued measurable function on $[0,1]$ and that $|f(x)-f(y)|$ is integrable on $[0,1]\times[0,1]$. Show that $f(x)$ is integrable on $[0,1]$.

Problem 2. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is absolutely continuous. Show that:

(a) f maps sets of measure zero to sets of measure zero
(b) f maps measurable sets to measurable sets.

Real Analysis, Optional Problems

Problem 3. Suppose that f is integrable on $\mathbb{R}^d$ and that

$$E_\alpha = \{x : |f(x)| > \alpha\}.$$

Prove that

$$\int_{\mathbb{R}^d} |f(x)|\,dx = \int_0^\infty m(E_\alpha)\,d\alpha.$$

Problem 4. Suppose $\{f_n\}$ is a sequence of integrable functions on $\mathbb{R}$ such that

$$\sum_{n=1}^\infty \int_{\mathbb{R}} |f_{n+1}-f_n| < \infty.$$

Show that there is a function f such that

$$\lim_{n\to\infty}\int_{\mathbb{R}} |f_n - f| = 0.$$

Complex Analysis, Mandatory Problems

Problem 1. Let $D(0,1) \subset \mathbb{C}$ denote the open unit disk. Let $f\colon D(0,1) \to D(0,1)$ be an analytic function. Suppose that $f(0) = f'(0) = 0$. Prove that

$$|f(z)| \leq |z|^2 \text{ and } |f''(0)| \leq 2$$

for all $z \in D(0,1)$.

Problem 2. Use the residue theorem to evaluate the integral

$$\int_0^\infty \frac{\sqrt{x}}{1+x^3}\mathrm{d}x.$$

Complex Analysis, Optional Problems

Problem 3. Let $D(0,1) \subset \mathbb{C}$ denote the open unit disk. Let $f\colon D(0,1) \to D(0,1)$ be an analytic function. Suppose that f has at least two fixed points (i.e. $f(x_1) = x_1$ and $f(x_2) = x_2$ for some $x_1, x_2 \in D(0,1)$ with $x_1 \neq x_2$). Prove that f must be the identity function.

Problem 4. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function such that $\lim_{z\to\infty} f(z) = \infty$. Show that f is a polynomial.

Be sure to justify your answers!