Preliminary Examination in Analysis

January 2022

Instructions

- This is a three-hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis.
- You should work problems from the section on advanced calculus and from the section of the option that you have chosen.
- You are to work a total of five problems (four mandatory problems and one optional problem).
- You must work two mandatory problems from each part.
- Please indicate clearly on your test paper which optional problem is to be graded.
- Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. Suppose that (S, d) is a metric space. Let $\{p_n\}$ be a sequence from S and suppose that $\sum_{n=1}^{\infty} d(p_n, p_{n+1})$ is finite. Prove that $\{p_n\}$ is Cauchy in S.

2. Suppose that f is nonnegative and continuous on $[a, b]$ and that

$$\int_a^b f(x)\,dx = 0.$$

Prove that $f(x) = 0$ on $[a, b]$.

Advanced Calculus, Optional Problems

3. (2^n Test) Suppose that $\{a_n\}$ is a nonincreasing sequence of nonnegative real numbers, i.e.,

$$a_1 \geq a_2 \geq \ldots \geq a_n \geq \ldots \geq 0$$

and that $\lim_{n\to\infty} a_n = 0$. Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{m=0}^{\infty} 2^m a_{2^m}$ converges.

4.

(a) Show that, if $\{f_n\}$ is a sequence of continuous functions and converges uniformly on $[a, b]$ to a function f, then f is also continuous on $[a, b]$.

(b) Give an example of a sequence $\{f_n\}$ of continuous functions on $[a, b]$ that converges pointwise to a function f so that f is not continuous.

Real Analysis, Mandatory Problems

1. Suppose that $E \subset \mathbb{R}^d$ is a measurable set. Let $h \in \mathbb{R}^d$, and define $E + h = \{e + h, e \in E\}$. Show that $E + h$ is measurable, and that $m(E + h) = m(E)$.

2. Let $f : \mathbb{R}^d \to \mathbb{R}$ be an integrable function. For every $\alpha > 0$, define $E_\alpha = \{x : |f(x)| > \alpha\}$.
   i) Show that E_α is measurable for all $\alpha > 0$.
   ii) Show that

$$\int_{\mathbb{R}^d} |f(x)|dx = \int_0^\infty m(E_\alpha)d\alpha.$$

Real Analysis, Optional Problems

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative and integrable function so that $\int f dx = 1$. Define

$$g(x) = \sum_{n=1}^\infty f(3^n x)$$

Compute $\int_{\mathbb{R}} g(x)dx$. Make sure to justify your steps!

4. Assume that $f : \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function, and that there is $M > 0$ so that $|f'(x)| \leq M$ for a.e. x. Let $g : [a, b] \to \mathbb{R}$ be a function of bounded variation. Show that $f \circ g : [a, b] \to \mathbb{R}$ is a function of bounded variation.

Complex Analysis, Mandatory Problems

1. Let $f(z)$ be an entire function. Suppose that the function $z \mapsto f(\overline{z})$ is also entire. Prove that $f(z)$ is constant.

2. Let $n \geq 2$ be a positive integer. Use the residue theorem to evaluate the integral

$$\int_0^\infty \frac{1}{1+x^n} \mathrm{d}x.$$

Hint: Use a wedge of angle $2\pi/n$.

Complex Analysis, Optional Problems

3. Let f be an entire function. Suppose that there exists a positive integer n such that

$$|f(z)| \geq |z|^n \text{ for all } |z| \geq 2022.$$

Prove that f is a polynomial and that its degree is at least n.

4. Let $D = \{z \in \mathbb{C} : |z| < 1\}$, and let $f : D \to D$ be an analytic function. Suppose that there exists $a \in D \setminus \{0\}$ such that $f(a) = f(-a) = 0$. Prove that $|f(0)| \leq |a|^2$. What can you conclude if $|f(0)| = |a|^2$?