Preliminary Examination in Analysis

January 2023

Instructions

- This is a three-hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis.
- You should work problems from the section on advanced calculus and from the section of the option that you have chosen.
- You are to work a total of five problems (four mandatory problems and one optional problem). Each problem is of equal value.
- You must work two mandatory problems from each part.
- Please indicate clearly on your test paper which optional problem is to be graded.
- Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function such that f' is continuous on $[0, 1]$. Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \varepsilon$$

whenever $|x - y| < \delta$ with $x, y \in [0, 1]$.

2. Let $A, B \subset \mathbb{R}$ be bounded sets. Define

$$A + B := \{a + b : a \in A, b \in B\} \quad \text{and} \quad A - B := \{a - b : a \in A, b \in B\}.$$

For each of the following statements, prove or exhibit a counterexample:

a) $\sup(A + B) = \sup A + \sup B$;

b) $\sup(A - B) = \sup A - \sup B$.

Advanced Calculus, Optional Problems

3. a) State the definition of a metric space X.

b) Show that if X is compact and $f : X \to \mathbb{R}$ is continuous, then f is uniformly continuous on X.

4. Suppose that $a_0, a_1, \ldots, a_n$ are real numbers with

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \ldots + \frac{a_n}{n+1} = 0.$$

Show that the polynomial $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ has a root in the interval $(0, 1)$.

Real Analysis, Mandatory Problems

1. a) Let E be a subset of $\mathbb{R}^d$. State the definition of the outer measure $m_*(E)$.
   b) Let E_1, E_2 be two subsets of $\mathbb{R}^d$ with $\text{dist}(E_1, E_2) > 0$. Show that

$$m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2).$$

2. Suppose $f : [0, \infty) \to \mathbb{R}$ is continuous and integrable on $\mathbb{R}$. Show that

$$\lim_{n\to\infty} \int_0^\infty f(x^n)\, dx = f(0).$$

Hint: Analyze the integral from 0 to 1 and the integral from 1 to ∞ separately.

Real Analysis, Optional Problems

3. Let $f(x) = \frac{\sin x}{x}$ for $x \neq 0$. Show that the limit

$$\lim_{N\to\infty} \int_{-N}^{N} f(x)\, dx$$

exists and is finite, but f is not a Lebesgue integrable function on $\mathbb{R}$.

4. Suppose $f \in L^1(\mathbb{R}^n)$. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that for all measurable sets $E \subset \mathbb{R}^n$ with $m(E) < \delta$,

$$\int_E |f(x)|\, dx < \epsilon.$$

You may assume continuous compactly supported functions are dense in $L^1(\mathbb{R}^n)$.

Complex Analysis, Mandatory Problems

1. Let f be a holomorphic function on $\mathbb{C}\setminus\{0\}$ such that $\lim_{z\to\infty} f(z) = A$. Let γ be the circle $|z| = 1$. Prove that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z}\mathrm{d}z = A.$$

2. Use complex analysis to evaluate the integral

$$\int_0^{2\pi} \frac{\mathrm{dx}}{3-\cos x}.$$

(Hint: Make the substitution $z = e^{ix}$.)

Complex Analysis, Optional Problems

3. Let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic function whose only singularity is a simple pole at $z = 1$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of f around 0. Prove that $a_n \neq 0$ for all large enough n.

4. Prove or exhibit a counterexample: If $f : \mathbb{C} \to \mathbb{C}$ is an entire function that maps every unbounded sequence to an unbounded sequence, then f is a polynomial.