Preliminary Examination in Analysis

January 2024

Instructions

- This is a three-hour examination on Advanced Calculus and Real Analysis.
- You are to work a total of five problems (four mandatory problems, two from each section, and one optional problem).
- You must work the two mandatory problems from each part.
- Please indicate clearly on your test paper which optional problem is to be graded.
- Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. Suppose $\{a_n\}$ and $\{b_n\}$ are two complex sequences such that

$$\lim_{n\to\infty} a_n b_n = 0$$

Show that at least one of $\{a_n\}$ and $\{b_n\}$ has a subsequence that converges to zero.

2. Suppose $f : [a, b] \to \mathbb{R}$ is bounded, and moreover f is Riemann integrable on $[a, c]$ for all $a < c < b$. Show that f is Riemann integrable on $[a, b]$.

Advanced Calculus, Optional Problems

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $\lim_{x\to\infty} f'(x) = 0$. Show that if the sequence $\{f(n)\}_{n\in\mathbb{N}}$ converges, then the limit $\lim_{x\to\infty} f(x)$ exists.

4. A function $f : \mathbb{R} \to \mathbb{R}$ is called Lipschitz if there exists $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in \mathbb{R}$. Show that every Lipschitz function on $\mathbb{R}$ is uniformly continuous on $\mathbb{R}$, but not every uniformly continuous function on $\mathbb{R}$ is necessarily Lipschitz on $\mathbb{R}$.

Real Analysis, Mandatory Problems

For a measurable subset E of $\mathbb{R}^d$, we use $m(E)$ to denote the Lebesgue measure of E.

1. Let $f = f(x, y)$ be a real-valued, continuous function on

$$S = \{(x, y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$$

and

$$F(x) = \int_0^1 f(x, y)\, dy.$$

Show that if $g(x, y) = \frac{\partial f}{\partial x}(x, y)$ is continuous on S, then $F(x)$ is differentiable on $(0, 1)$ and

$$F'(x) = \int_0^1 g(x, y)\, dy.$$

2. Let E be a subset of $\mathbb{R}$ with measure zero. Show that the set

$$\{x^2 : x \in E\}$$

also has measure zero.

Real Analysis, Optional Problems

3. (a). State Fatou's Lemma.
   (b). State the Monotone Convergence Theorem.
   (c). Use Fatou's Lemma to prove the Monotone Convergence Theorem.

4. Let f be an integrable function on $\mathbb{R}^d$ and

$$E_n = \{x \in \mathbb{R}^d : |f(x)| > n\}.$$

Show that

$$\lim_{n\to\infty} n \cdot m(E_n) = 0.$$