Preliminary Examination in Analysis

January 2025

Instructions

- This is a three-hour exam on Advanced Calculus and Real or Complex Analysis.
- Please work a total of five problems (four mandatory problems, two from each section, and one optional problem). You *must* work the mandatory problems from each part
- Please indicate clearly on your test paper which optional problem is to be graded
- Please indicate clearly what theorems and definitions you are using

Advanced Calculus, Mandatory Problems

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is positive, bounded, and Riemann integrable on $[0, 1]$. Show that f^2 is Riemann integrable on $[0, 1]$.

2. Suppose $\{a_n\}$ is a nonnegative summable sequence and $g : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $g'(0) = g(0) = 0$. Show that $\sum_{n=1}^{\infty} g(a_n)$ converges.

   Hint: Consider $g(h)/h$.

Advanced Calculus, Optional Problems

3. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous. Show that for any $\epsilon > 0$ there exists $M > 0$ such that $|f(x) - f(y)| < M|x - y| + \epsilon$ for all $x, y \in [0, 1]$.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that f' is continuous on $[0, 1]$. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in [0, 1]$ with $|x - y| < \delta$,

$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \varepsilon$$

Real Analysis, Mandatory Problems

1. Suppose that $\{f_n\}$ is a monotone nondecreasing sequence of measurable functions with $\lim_{k\to\infty} f_k(x) = f(x)$ for all x. Using the definition of measurability, give a direct proof that the function f is measurable.

2. Recall that the Fourier transform $\widehat{f}$ of a function $f \in L^1(\mathbb{R})$ is

$$\widehat{f}(\xi) = \int e^{-i\xi x} f(x)\, dx.$$

   (a) Show that $\widehat{f}$ is continuous on $\mathbb{R}$.

   (b) Show that $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$. For this part, you may assume that the set of all linear combinations of characteristic functions of bounded open intervals is dense in $L^1(\mathbb{R})$.

Real Analysis, Optional Problems

3. Suppose that f is an integrable function on $\mathbb{R}^d$ and that

$$E_\alpha = \{x : |f(x)| > \alpha\}.$$

   Prove that

$$\int_{\mathbb{R}^d} |f(x)|\, dx = \int_0^\infty m(E_\alpha)\, d\alpha.$$

4. This problem concerns real-valued measurable functions $f : E \subset \mathbb{R} \to \mathbb{R}$.

   (a) State Egoroff's Theorem.

   (b) Suppose that $E \subset \mathbb{R}$ is a measurable set with $m(E) < \infty$, and that $\{f_n\}_{n=1}^\infty$ is a sequence of uniformly bounded, real-valued measurable functions on E with $f_n \to f$ pointwise for a.e. x. Using Egoroff's Theorem, prove that

$$\lim_{n\to\infty} \int_E f_n(x)\, dx = \int_E f(x)\, dx.$$

Complex Analysis, Mandatory Problems

1. Let $\{f_n\}$ be a sequence of analytic functions on a region $\mathcal{A} \subset \mathbb{C}$ converging uniformly on compact subsets of $\mathcal{A}$.

   (a) Prove that a limit function f exists that is analytic on $\mathcal{A}$.

   (b) Assume that f is not identically zero. Then, there exists a point $z_0 \in \mathcal{A}$ with $f(z_0) = 0$ if and only if there is a sequence $z_n \to z_0$ in $\mathcal{A}$ so that $f_n(z_n) = 0$ for all n sufficiently large. HINT: Apply Rouché's Theorem.

2. Compute the following integral

$$\int_0^\infty \frac{x \sin(x)}{(x^2+1)^2}\, dx.$$

   Justify all steps of the calculation.

Complex Analysis, Optional Problems

3. Let f be analytic on the unit disk $\mathbb{D}$ and continuous on its closure $\overline{\mathbb{D}}$. Suppose that $|f(z)| = 1$ for all $|z| = 1$.

   (a) Prove that if f never vanishes in $\mathbb{D}$, then f is a constant.

   (b) Prove that there are only finitely many zeros of f in $\mathbb{D}$.

   (c) Suppose $a_1, a_2, \ldots, a_n$ are the zeros of f in $\mathbb{D}$. Prove that there is an angle $\theta \in \mathbb{R}$ so that

$$f(z) = e^{i\theta} \left(\frac{z - a_1}{1 - \overline{a}_1 z} \right) \cdots \left(\frac{z - a_n}{1 - \overline{a}_n z} \right).$$

4. Find all entire functions $f(z)$ satisfying the bound

$$|f(z)| \geq |z|,$$

   for all $z \in \mathbb{C}$.