6/8/p1

ANALYSIS PRELIMINARY EXAM

June 2, 2008

Instructions

1. This is a three hour examination which consists of two parts: Advances Calculus and Real *or* Complex Analysis. You should work problems from section on Advanced Calculus and from the section of the option that you have chosen.

2. You need to work a total of 5 problems, four mandatory problems (two mandatory problems from each part), and one optional problem from either advanced calculus or the option you have chosen. Please indicate clearly on your test paper whether you are taking the Real or Complex Analysis option, and which optional problems you are solving.

3. Do not put your name on any sheet except the cover page. The papers will be blind-graded.

4. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problems than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

6/4/p2

Advanced Calculus, Mandatory Problems

1. Let $\{x_k\}_{k=1}^{\infty}$ be a convergent sequence in a metric space (X, d), with $x_0 = \lim_{k\to\infty} x_k$. Show that the set $\{x_0, x_1, x_2, \ldots\}$ is a compact set in (X, d).

2. For each pair of natural numbers (n, m), let $a_{nm} \in \mathbf{R}$. Suppose that for each m, we have

$$\lim_{n\to\infty} a_{nm} = 1.$$

If we fix n, do we have

$$\lim_{m\to\infty} a_{nm} = 1?$$

(Give a proof, if you think it is true. Otherwise, give a counterexample)

Advanced Calculus, Optional Problems

1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers which converges to a finite limit L. Set $y_n = \inf\{x_k : k \geq n\}$. Show that $L = \sup\{y_n : n \geq 1\}$.

2. Suppose that f is a continuous real-valued function defined on the interval $(-1, 1)$. If f is differentiable except possibly at 0 and $\lim_{x\to 0} f'(x)$ exists, does it follow that f is differentiable at 0?

Real Analysis, Mandatory Problems

1. For a given Lebesgue measurable set $E \subseteq \mathbf{R}^n$, answer the following two questions:
(a) State the definition that $f : E \to \mathbf{R}$ is Lebesgue measurable.
(b) Suppose that $|E| < +\infty$ and $f : E \to \mathbf{R}$ is Lebesgue measurable, prove that for any $\epsilon > 0$ there exists a closed subset $F \subseteq E$ such that

$$|E \setminus F| < \epsilon, \quad \sup_{x \in F} |f(x)| < +\infty.$$

2. Given that $f \in L^1(\mathbf{R}^n)$, construct a family of functions $f_k \in L^1(\mathbf{R}^n)$ such that for each $k \geq 1$, f_k is bounded and has compact support, and

$$\lim_{k \to \infty} \int_{\mathbf{R}^n} |f_k(x) - f(x)| \, dx = 0.$$

Real Analysis, Optional Problems

1. Suppose that $f : \mathbf{R}^n \to \mathbf{R}$ satisfies

$$\int_{\mathbf{R}^n} |f(x)| \, dx = 0.$$

Prove that $f = 0$ a.e. on $\mathbf{R}^n$.

2. Suppose that $E \subseteq \mathbf{R}^n$ is Lebesgue measurable with $|E| < +\infty$, and $f \in L^1(E)$. Prove that for any $\epsilon > 0$ there exists a $\delta > 0$ depending on ϵ and f such that for any measurable subset $F \subseteq E$,

$$\int_F |f(x)| \, dx < \epsilon, \quad \text{whenever } |F| < \delta.$$

January 2008

Complex Analysis, Mandatory Problems

1. Calculate the integral

$$\int_0^\infty \frac{dx}{1+x^5}.$$

(Hint: Consider the path from 0 to R along the real axis, followed by the circular arc from R to $Re^{\frac{2\pi i}{5}}$, and the line segment from the latter point to the origin. Then let $R \to \infty$.)

2. How many zeros (counting multiplicity) does the polynomial

$$3z^9 + 8z^6 + z^5 + 2z^3 + 1$$

have in the annulus $\{z: \ 1 < |z| < 2\}$?

Complex Analysis, Optional Problems

1. Prove or disprove: If an entire function $f(z)$ maps every unbounded sequence to an unbounded sequence, then f is a polynomial.

2. Suppose that f is a complex valued function on the open unit disk $D = \{z: \ |z| < 1\}$ such that $g = f^2$ and $h = f^3$ are both analytic in D. Prove that f is also analytic in D.