6/9/p1

ANALYSIS PRELIMINARY EXAM

8 June 2009

NAME: ______________________________ OPTION: ________________

Instructions

1. This is a three hour examination which consists of two parts: **Advanced Calculus and Real** *or* **Complex Analysis**. You must work problems from the section on Advanced Calculus and from the section of the option, Reals or Complex, that you have chosen.

2. You need to work a total of 5 problems: Four mandatory problems (two mandatory problems from each of two parts), and one optional problem from either advanced calculus or the option you have chosen. Please indicate clearly on your test paper whether you are taking the **Real or Complex Analysis option**, and which optional problem you are solving. **Only 5 problems will be graded.**

3. Do not put your name on any sheet except the cover page. The papers will be blind-graded.

4. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to partial solutions of the easy parts of two different problems. Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. (a) State the definition of a compact set in a metric space.

   (b) Suppose that f is a continuous map from a compact metric space X into a metric space Y. Show that $f(X)$ is a compact set.

2. (a) Say what it means for a function f on the interval $[a, b]$ to be Riemann integrable on $[a, b]$.

   (b) Suppose that $f(x) \geq 0$, that f is continuous, and that $\int_a^b f = 0$. Using the definition of Riemann integrability, show that $f(x) = 0$ on $[a, b]$.

Advanced Calculus, Optional Problems

1. Suppose that $\{f_n\}$ is a sequence of continuous functions on $[0, 1]$, and that f is another function on $[0, 1]$ with the property that $f_n \to f$ uniformly for $x \in [0, 1]$. Show that f is continuous on $[0, 1]$.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and for any $x, y \in \mathbb{R}$ and a fixed θ with $0 < \theta < 1$,

$$|f(x) - f(y)| < \theta \, |x - y| \, .$$

Fix $x_0 \in \mathbb{R}$ and define a sequence on $\mathbb{R}$ by

$$x_1 = f(x_0)$$

and

$$x_{n+1} = f(x_n).$$

   (a) Show that $\{x_n\}$ is a Cauchy sequence. (*Hint*: Consider $x_{n+1} - x_n$).

   (b) Let x be the limit of the sequence $\{x_n\}$. Show that $f(x) = x$.

Real Analysis, Mandatory Problems

1. Let $f : \mathbb{R}^d \to \mathbb{R}$ be Lebesgue measurable and define

$$f_n(x) = \begin{cases} f(x) & |f(x)| \le n \\ 0 & |f(x)| > n. \end{cases}$$

   (a) Prove that

$$\lim_{n\to\infty} \int_{R^d} |f_n|^2 = \int_{\mathbb{R}^d} |f|^2.$$

   (b) If $|f|^2 \in L^1(\mathbb{R}^d)$, then prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |f_n - f|^2 = 0.$$

2. (a) Let $E \subset \mathbb{R}^d$ be a compact, Lebesgue measurable set with $m(E) > 0$. Prove that there exists $x \in E$ so that for any $\delta > 0$ one has $m(E \cap B_\delta(x)) > 0$, where $B_R(y)$ is the ball of radius $R > 0$ centered at $y \in \mathbb{R}^d$.

   (b) Let $-\infty < a < b < \infty$ and consider a closed, measurable set $F \subset [a, b]$ with $m(F) = 0$. Let χ_F be the characteristic function for F. Is χ_F Riemann integrable on $[a, b]$? Prove the statement if it is true for all such sets F, or give a counterexample if it is false.

Real Analysis, Optional Problems

1. Let $f_n \in AC([a, b])$ be a sequence of absolutely continuous functions on the bounded interval $[a, b] \subset \mathbb{R}$. Suppose that $\lim_{n\to\infty} f_n(a) = c$ exists and is finite. If $f_n' \in L^1([a, b])$ and $\{f_n'\}$ is L^1-Cauchy, then prove that the pointwise limit f of the sequence f_n exists, that $f_n \to f$ uniformly on $[a, b]$, and that $f \in AC([a, b])$.

2. (a) Let $E \subset \mathbb{R}^n$ be Lebesgue measurable with $m(E) = 0$. If $B \subset E$, then prove that B is Lebesgue measurable and $m(B) = 0$.

   (b) Suppose that $E \subset \mathbb{R}^1$ is Lebesgue measurable. Suppose there is a function $f : E \to \mathbb{R}$, with $f > 0$ almost everywhere on E, and $f \in L^1(E)$. Prove that if $\int_E f = 0$, then $m(E) = 0$.

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Complex Analysis, Mandatory Problems

1. Let $f(z)$ be a function that is analytic in $|z| < 1$ and suppose that

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\, d\theta \leq 1$$

whenever $0 < r < 1$. Show that $|f'(z)| \leq (1-|z|)^{-2}$ for $|z| < 1$.

2. Let D be the open right half-plane and let $f(z)$ be a bounded analytic function in D having a continuous extension to the imaginary axis. Show that if $|f(iy)| \leq 1$ for all real numbers y, then $|f(z)| \leq 1$ for all z in D. Show that the assumption that f is bounded on D cannot be omitted. (*Hint*: Consider $g(z) = f(z)/(1+z)^{\epsilon}$ on a large semidisc where $\epsilon > 0$.)

Complex Analysis, Optional Problems

1. Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{e^x + e^{-x}}\, dx = \frac{\pi}{e^{\pi/2} + e^{-\pi/2}}$$

by integrating the function $f(z) = \dfrac{e^{iz}}{e^z + e^{-z}}$ around the rectangle with vertices $-R$, R, $R + i\pi$, $-R + i\pi$ and estimating the integral on the vertical edges.

2. a) Find an explicit one-to-one analytic mapping of the complex plane with the interval $(-\infty, 0]$ removed onto the open unit disc. (*Hint*: First map to the right half-plane.)

   b) Use part (a) to show that the range of a non-constant entire function intersects the ray $\{re^{i\theta} : 0 \leq r < \infty\}$ for each value of θ.