Analysis Preliminary Examination

May 29, 2013

Instructions

1) This is a three-hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis. You should work problems from the section on advanced calculus and from the section of the option you have chosen.

2) You are to work a total of five problems – four mandatory problems (two mandatory problems from each part) and one optional problem you choose. **Please indicate clearly in your solutions which optional problem is to be graded.**

3) Give complete and detailed arguments. Indicate clearly the theorems and definitions you are using. Each question is weighted equally.

4) Be sure to write your solutions on only the front side of the paper provided (since only this side will be copied for the graders).

Advanced Calculus

Mandatory Problems

1. Let $\{f_k\}$ be a sequence of functions $f_k : \mathbb{R} \to \mathbb{R}$. Suppose that
   (a) $\{f_k\}$ converges uniformly to f on $\mathbb{R}$ and
   (b) each f_k is continuous at x_0.

Show that f is continuous at x_0.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0.$$

Show that f is uniformly continuous.

Optional Problems

3. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that

$$\lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n) = \infty.$$

Show that for any complex sequence $\{z_n\}_{n=1}^{\infty}$ with limit z,

$$\lim_{n\to\infty} \frac{a_1 z_1 + a_2 z_2 + \cdots a_n z_n}{a_1 + a_2 + \cdots + a_n} = z.$$

4. Let $f(x, y)$ be a real-valued, continuous function on

$$S = \{(x, y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$$

and put

$$F(x) = \int_0^1 f(x, y)\, dy.$$

Show that if $g(x, y) = \frac{\partial f}{\partial x}(x, y)$ is continuous on S, then $F(x)$ is differentiable on $(0, 1)$ and

$$F'(x) = \int_0^1 g(x, y)\, dy$$

for every x in $(0, 1)$. Hint: Use uniform continuity of g on S and the mean value theorem.

Real Analysis

Mandatory Problems

1. Let $\{E_k\}_{k=1}^{\infty}$ be a sequence of measurable subsets of $\mathbb{R}$ and let

$$E = \{x \in \mathbb{R} : \ x \in E_k \text{ for infinitely many } k\}.$$

Show that if

$$\sum_{k=1}^{\infty} m(E_k) < \infty,$$

then $m(E) = 0$.

2. Let $E \subseteq \mathbb{R}$ be measurable and $0 < m(E) < \infty$. Prove that there exists a measurable set $F \subseteq E$ such that $m(F) = m(E)$ and

$$m(E \cap B(x,r)) > 0 \quad \text{for every } x \in F \text{ and every } r > 0.$$

Optional Problems

3. A function $f : \mathbb{R} \to [-\infty, \infty]$ is said to be measurable if the set $\{x \in \mathbb{R} : \ f(x) > \alpha\}$ is measurable for every $\alpha \in \mathbb{R}$. Use this definition to show that if $\{f_k\}$ is a sequence of measurable functions with $\lim_{k\to\infty} f_k(x) = f(x)$ for every $x \in \mathbb{R}$, then f is measurable.

4. Let $f(x,y)$ be a nonnegative measurable function on $\mathbb{R}^2$. Suppose that for a.e. $x \in \mathbb{R}$, $f(x,y)$ is finite for a.e. y. Show that for a.e. $y \in \mathbb{R}$, $f(x,y)$ is finite for a.e. x.

Complex Analysis

Mandatory Problems

1. Let D be a convex domain containing the origin and let f be a continuous complex-valued function on D such that $\int_{\Delta} f(w)\, dw = 0$ for every triangle Δ in D. Given z in D, define

$$F(z) = \int_{\gamma(z)} f(w)\, dw,$$

where $\gamma(z)$ is the straight line from the origin to z. Give a direct elementary proof that F is holomorphic in D with $F' = f$ in D.

2. Compute the integral $\int_0^{\infty} \frac{x^3}{1+x^5}\, dx$ by applying the residue theorem for a circular sector with center at the origin. Your solution should obtain the limit of an integral along the circular boundary of your sector as the radius of the sector approaches infinity.

Optional Problems

3. Show that if f is an entire function with $\lim_{z\to\infty} |f(z)| = \infty$ then f is a polynomial.

4. Suppose D is an open set in the complex plane with the property that every holomorphic function on D has a holomorphic antiderivative in D. Show that every real harmonic function in D is the real part of a holomorphic function in D.