Preliminary Examination in Analysis

June 2014

Instructions

- This is a three-hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis.
- You should work problems from the section on advanced calculus and from the section of the option you have chosen.
- You are to work a total of five problems (four mandatory problems and one optional problem).
- You must work two mandatory problems from each part.
- Please indicate clearly on your test paper which optional problem is to be graded.
- Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. Suppose that f is Riemann integrable on $[a, b]$ for every $b > a$. We define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx,$$

provided the limit exists, in which case we say that the integral converges. Prove that if $f(x) \geq 0$ and $f(x)$ is monotonically decreasing for $x \geq 1$, then

$$\int_1^\infty f(x)\,dx$$

converges if and only if

$$\sum_{n=1}^\infty f(n)$$

converges.

2. Suppose that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space (X, d). Show that the sequence $\{d(p_n, q_n)\}$ converges.

Advanced Calculus, Optional Problems

3. Prove that for $R > 0$,

$$\int_0^\pi e^{-R\sin\theta}\,d\theta < \frac{\pi}{R}.$$

Hint: First obtain a lower estimate for $\sin\theta$ when $0 \leq \theta \leq \pi/2$.

4. Let $\{a_n\}$ be a sequence of real numbers and define sequences $\{s_n\}_{n=0}^\infty$ and $\{\sigma_n\}_{n=0}^\infty$ by

$$s_n = a_0 + a_1 + \cdots a_n, \quad \sigma_n = \frac{s_0 + s_1 + \cdots s_n}{n}.$$

Suppose that $\lim_{n\to\infty} \sigma_n = \sigma$ and let $f(x) = \sum_{n=0}^\infty a_n x^n$.

a) Show that if $|x| < 1$, then $f(x)$ converges and

$$f(x) - \sigma = (1-x)^2 \sum_{n=0}^\infty (n+1)(\sigma_n - \sigma)x^n.$$

b) Show that $\lim_{x\to 1^-} f(x) = \sigma$.

Real Analysis, Mandatory Problems

1. Let f, g be two real-valued integrable functions on $\mathbb{R}$. Show that if
$$\int_E f(x)\,dx = \int_E g(x)\,dx \quad \text{for every measurable } E \subset \mathbb{R},$$
then $f(x) = g(x)$ a.e.

2. Let f be an integrable function on $\mathbb{R}$. Define
$$f_n(x) = \begin{cases} f(x) & \text{if } |x| < n \text{ and } |f(x)| < n, \\ 0 & \text{otherwise .} \end{cases}$$
Show that
$$\int_{\mathbb{R}} |f_n(x) - f(x)|\,dx \to 0 \quad \text{as } n \to \infty.$$

Real Analysis, Optional Problems

3. Show that there exists a sequence of integrable functions f_k on $[0,1]$ such that
$$\int_0^1 |f_k(x)|\,dx \to 0 \quad \text{as } k \to \infty,$$
but $f_k(x) \to 0$ for no $x \in [0,1]$.

4. Consider the function defined on $\mathbb{R}$ by
$$f(x) = \begin{cases} \sqrt{x} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise .} \end{cases}$$
For a fixed enumeration $\{r_n\}$ of the rationals $\mathbb{Q}$, let
$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n).$$
Prove that $F(x)$ is finite for a.e. $x \in \mathbb{R}$.

Complex Analysis, Mandatory Problems

1. Prove that if f is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \leq AR^k + B$$

for all $R > 0$, and for some integer $k \geq 0$ and some constants A, $B > 0$, then f is a polynomial of degree $\leq k$.

2. Show that if $a > 0$, then

$$\int_0^\infty \frac{\log x}{x^2 + a^2}\, dx = \frac{\pi}{2a} \log a.$$

Complex Analysis, Optional Problems

3. Let f be holomorphic in the disc $|z| < 1$ with $f'(0) \neq 0$. Show that there exists some circle C inside this disc with

$$\int_C \frac{f'(0)}{f(z) - f(0)}\, dz = 2\pi i.$$

4. Let q be a polynomial of degree m with distinct roots $z_1, \ldots, z_m$ and let p be a polynomial of degree at most $m - 2$, where $m \geq 2$. Let C_R be the circle having counterclockwise orientation with center at the origin and sufficiently large radius R. Show that

a)

$$\int_{C_R} \frac{p(z)}{q(z)}\, dz = 0.$$

b)

$$\sum_{k=1}^{m} \frac{p(z_k)}{q'(z_k)} = 0.$$