ANALYSIS PRELIMINARY EXAM
June 3, 2015

Instructions

- This is a three hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis.
- You should work problems from the section on advanced calculus and from the section of the option you have chosen.
- You are to work a total of five problems (four mandatory problems and one optional problem).
- Please indicate clearly on your test paper which optional problem is to be graded.
- Indicate clearly what theorems and definitions you are using.

In what follows $\mathbb{R}^n$ denotes Euclidean n space, while $\mathbb{R}, \mathbb{C}$ denote the real and complex numbers.

ADVANCED CALCULUS
MANDATORY PROBLEMS

1. Given two nonempty sets in $\mathbb{R}^n$, the distance between C and D is defined by

$$d(C, D) = \inf\{|x - y| : x \in C, y \in D\}.$$

   (a) If $a \in \mathbb{R}^n$ and D is closed, prove that there exists a $d \in D$ such that $d(\{a\}, D) = |a - d|$.

   (b) If C is compact and D is closed, prove that there exist $c \in C$ and $d \in D$ such that $d(C, D) = |c - d|$.

2. Let $f : [0, 1] \rightarrow \mathbb{R}$ be defined by $\begin{cases} f(x) = 0, & \text{for irrational } x \\ f(x) = 1/n & \text{for rational } x = m/n \end{cases}$
where m, n are nonnegative integers with no common factors. Prove that f is Riemann integrable on [0,1] and find the value of $\int_0^1 f(x)dx$.

OPTIONAL PROBLEMS

3. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Prove that for each $\varepsilon > 0$, there is an $M > 0$ such that

$$|f(x) - f(y)| \leq M\,|x - y| + \varepsilon \qquad \text{for all } x, y \in [0, 1].$$

4. Let $f : [a, b] \to \mathbb{R}$ be a continuous function that satisfies

$$\int_a^b x^n f(x)\, dx = 0 \qquad \text{for each nonnegative integer } n.$$

   Prove that $f(x) = 0$ for each $x \in [a, b]$.

REAL ANALYSIS
MANDATORY PROBLEMS

1. Let $\{E_n\}_{n=1}^{\infty}$ be a countable family of measurable subsets of $\mathbb{R}^2$ and let

$$E = \{x \in \mathbb{R}^2 : \; x \in E_n \text{ for infinitely many positive integers } n\}.$$

   (a) Show that E is measurable.

   (b) Show that if the series $\sum_{n=1}^{\infty} m(E_n)$ converges then $m(E) = 0$. Here m denotes Lebesgue measure.

2. Let $E = \mathbb{R}^2$ and suppose f is integrable on E.

   (a) Show that $f(x)$ is finite for almost every x in E.

   (b) Apply a convergence theorem to show that for every $\epsilon > 0$ there exists a bounded integrable function g on E with compact support and satisfying

$$\left| \int_E f - \int_E g \right| < \epsilon.$$

OPTIONAL PROBLEMS

3. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions on a measurable set E in $\mathbb{R}^2$ such that f_1 is integrable on E and $\sum_{n=1}^{\infty} \int_E |f_{n+1} - f_n| < \infty$. Show $\{f_n\}$ converges almost everywhere to an integrable function f on E and $\lim_{n\to\infty} \int_E f_n = \int_E f$.

4. Show that if f is a real-valued function that is both absolutely continuous and strictly increasing on an interval $[a, b]$, then

$$\int_U f' = m(f(U))$$

for any open subset U of $[a, b]$.

COMPLEX ANALYSIS
MANDATORY PROBLEMS

1. Use Cauchy's theorem for derivatives or the residue theorem to verify for n a positive integer that

$$\int_0^\pi \sin^{2n}\theta \, d\theta = \pi \frac{(2n)!}{2^{2n}(n!)^2} .$$

2. Prove that if f is a univalent mapping of $B(0,1) = \{z : |z| < 1\}$ onto $B(0,1)$ then f is a Möbius or linear fractional transformation.

OPTIONAL PROBLEMS

3. Find the univalent function f which maps $B(0,1) = \{z : |z| < 1\}$ onto $\mathbb{C} \setminus (-\infty, -1/4]$ and satisfies $f(0) = 0, f'(0) = 1$.

4. Given $P(z) = z^7 + z^4 + 5z^3 + z + 1$ for $z \in \mathbb{C}$.

   (a) Determine how many zeros P has (counted according to multiplicity) in $B(0,1) = \{z : |z| < 1\}$.

   (b) Determine how many zeros P has (counted according to multiplicity) in $B(0,2) = \{z : |z| < 2\}$.