Preliminary Examination in Analysis

June 2016

Instructions

- This is a three-hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis.
- You should work problems from the section on advanced calculus and from the section of the option you have chosen.
- You are to work a total of five problems (four mandatory problems and one optional problem).
- You must work two mandatory problems from each part.
- Please indicate clearly on your test paper which optional problem is to be graded.
- Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. Suppose that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space (X, d). Show that the sequence $\{d(p_n, q_n)\}$ converges. Note that X is not assumed to be complete.

2. Let X be a metric space, and let $\{f_n\}$ be a sequence of real-valued functions on X.

   (a) Say what it means for the sequence $\{f_n\}$ to converge uniformly to a function f on X.

   (b) Suppose that $\{f_n\}$ converges uniformly to f on X. Let $p \in X$. Using the definition from part (a), prove that, if each f_n is continuous at p, then f is also continuous at p.

Advanced Calculus, Optional Problems

3.

   (a) Say what it means for a function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}$ to be uniformly continuous.

   (b) Suppose that f is a real-valued continuous function on $\mathbb{R}$ and that

$$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 0.$$

   Prove that f is uniformly continuous.

4.

   (a) State the Weierstrass approximation theorem for continuous functions on $[0, 1]$.

   (b) Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and that $\int_0^1 x^n f(x)\, dx = 0$ for all $n = 0, 1, 2, \cdots$. Prove that f is the zero function. You may assume the following theorem: if g is continuous and nonnegative on $[0, 1]$ and $\int_0^1 g(x)\, dx = 0$ (Riemann integral), then g is the zero function.

Real Analysis, Mandatory Problems

1. Let f be a Lebesgue integrable function in $\mathbb{R}^d$. Suppose that

$$\int_E f \geq 0$$

for every Lebesgue measurable set E in $\mathbb{R}^d$. Show that $f \geq 0$ a.e. in $\mathbb{R}^d$.

2.

(a) State Egorov's Theorem.
(b) State the Bounded Convergence Theorem.
(c) Use Egorov's Theorem to prove the Bounded Convergence Theorem.

Real Analysis, Optional Problems

3. Let f be uniformly continuous on $\mathbb{R}$. Suppose that f is Lebesgue integrable on $\mathbb{R}$. Show that

$$\lim_{|x|\to\infty} f(x) = 0.$$

4. A map $T : \mathbb{R}^d \to \mathbb{R}^d$ is called Lipschitz if there exists $M > 0$ such that

$$|T(x) - T(y)| \leq M|x - y| \quad \text{for any } x, y \in \mathbb{R}^d.$$

Show that if E is a subset of $\mathbb{R}^d$ with $m(E) = 0$ and $T : \mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz map, then $T(E)$ is Lebesgue measurable and $m(T(E)) = 0$.

Complex Analysis, Mandatory Problems

1. Let $|\alpha| < 1$ and set $f(z) = \dfrac{z-\alpha}{1-\bar{\alpha}z}$. Show that
   (a) $|f(z)| = 1$ on $|z| = 1$,
   (b) f is analytic in $|z| < 1$,
   (c) f maps the closed unit disk in a one-to-one fashion, onto itself.

2. Use the residue theorem to verify that

$$\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}\, dx = \frac{\pi}{2}.$$

Your solution should define the curve of integration you use and should verify any assertion you make about an integral approaching 0.

Complex Analysis, Optional Problems

3. Let f be a function analytic in $|z| < 1$, continuous on $|z| \le 1$, with the property that $|f(z)| = 1$ for all z on the boundary $|z| = 1$.

(a) Suppose that f has no zeros in the interior $|z| < 1$. Prove that f is identically constant. *Hint*: Apply the maximum principle to both f and $1/f$.

(b) Prove that, in general, f is a rational function. *Hint*: Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the zeros of f in $|z| < 1$, counting multiplicity, and consider the function

$$f(z) \prod_{j=1}^{n} \left(\frac{z-\alpha_j}{1-\bar{\alpha}_j z} \right)^{-1}$$

(see Problem 1).

4. Suppose that a function f is defined and analytic in the entire complex plane $\mathbb{C}$, and that for each point $z_0 \in \mathbb{C}$ at least one coefficient c_n in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

is equal to 0. Prove that f is a polynomial.
Hint: For each $k = 1, 2, 3 \ldots$ consider the set $E_k = \{z \in \mathbb{C} : f^{(k)}(z) = 0\}$, and argue that for some k the set E_k is *uncountable*, etc.