# Preliminary Examination in Analysis

## May 31, 2017

### Instructions

This is a three-hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis.

- You should work problems from the section on advanced calculus and from the section of the option you have chosen.
- You are to work a total of five problems: *four* mandatory problems and *one* optional problem.
- You must work *two* mandatory problems from each part.

Please be sure to indicate clearly on your test paper which optional problem is to be graded. In your solutions, please indicate clearly what theorems and definitions you are using.

## Advanced Calculus, Mandatory Problems

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that

$$\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = 0.$$

Prove that $f$ is uniformly continuous.

2. Suppose $\{I_k\}_{k \in \mathbb{N}}$ is a sequence of closed intervals $I_k = [a_k, b_k]$, such that $I_{k+1} \subseteq I_k$. Show that

$$\bigcap_{k=1}^{\infty} I_k$$

is non-empty. Must this be true if the $I_k$ are open?

## Advanced Calculus, Optional Problems

3. Suppose $\{a_n\}$ is a sequence of positive numbers such that the limit

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$$

exists and is less than 1. Show *directly* that

$$\sum_{n=1}^{\infty} a_n$$

converges (your solution should not simply appeal to the ratio test).

4. Consider the sequence defined recursively by $a_n = 1$ and $a_{n+1} = \sqrt{2a_n}$. Show that this sequence converges, and determine its limit. *Hint*: Can you show that $\{a_n\}$ is bounded above?

## Real Analysis, Mandatory Problems

1. Let $f(x, y)$ be a continuous function in $\mathbb{R}^2$. Assume that $\dfrac{\partial f}{\partial x}$ exists and is continuous in $\mathbb{R}^2$. Let

$$g(x) = \int_0^1 f(x, y)\, dy.$$

Show that $g$ is differentiable and

$$g'(x) = \int_0^1 \frac{\partial f}{\partial x}(x, y)\, dy.$$

2. Let $\{f_k\}$ be a sequence of measurable functions in $\mathbb{R}^d$. Suppose that

$$g(x) = \lim_{k \to \infty} f_k(x) \text{ exists for a.e. } x \in \mathbb{R}^d.$$

Show that $g$ is measurable in $\mathbb{R}^d$.

## Real Analysis, Optional Problems

3. Suppose that $f$ is a continuous real-valued function and consider the curve $\Gamma \subset \mathbb{R}^2$ given by $\{(x, f(x)) : x \in \mathbb{R}^d\}$. Prove that $\Gamma$ has measure zero as a subset of $\mathbb{R}^2$.

4. Let $E \subset \mathbb{R}^d$ be measurable and $0 < m(E) < \infty$. Prove that there exists a measurable set $F \subset E$ such that $m(F) = m(E)$, and that

$$m(E \cap B(x, r)) > 0 \quad \text{for any } x \in F \text{ and any } r > 0.$$

## Complex Analysis, Mandatory Problems

1. Show that if $f$ is an entire function and if the range of $f$ does not intersect the circle $|z| = 1$, then $f$ is constant.

2. Compute the integral below using the residue theorem with integration around a semicircle with its flat side placed symmetrically along the real axis.

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+1)^2}\, dx.$$

## Complex Analysis, Optional Problems

3. Show that there is no function $f$ analytic in the domain $D = \{z \in \mathbb{C} : z \neq 0\}$ such that $f'(z) = 1/z$ for all $z \in D$.

4. Let $f(z) = \sum_{n=0}^{\infty} z^{n!}$.
   (a) Show that $f(z)$ is defined and analytic on the disc $|z| < 1$.
   (b) Show that if $\lambda$ is a root of unity then $|f(r\lambda)| \to \infty$ as $r \to 1^-$.