Preliminary Examination in Analysis

June 2018

Instructions

This is a three-hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis.

You should work on problems from the section on advanced calculus and from the section of the option you have chosen.

You are to work a total of **five problems**: *four* mandatory problems and *one* optional problem. You must work *two* mandatory problems from each part.

Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

Problem 1. Suppose $A, B \subset \mathbb{R}$ are nonempty sets which are bounded above. Let

$$(A+B) = \{a+b | a \in A, b \in B\}.$$

Show that

$$\sup(A+B) = \sup A + \sup B.$$

Problem 2. Suppose f is Riemann integrable on $[a,b]$ so that there is a function g continuous on $[a,b]$ with $f = g'$ for all $x \in (a,b)$. Show that

$$\int_a^b f(x)\,dx = g(b) - g(a).$$

Note that since f is not necessarily continuous, one cannot simply apply the First Fundamental Theorem of Calculus. The hint is to pick a partition of the interval and apply the Mean Value Theorem to g on the subintervals of the partition.

Advanced Calculus, Optional Problems

Problem 3. Suppose f is differentiable on $\mathbb{R}$. We say that f has the reverse mean value property if for every $c \in \mathbb{R}$, there exists $a, b \in \mathbb{R}$ with $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Is it true that every continuously differentiable function has the reverse mean value property? Justify your answer.

Problem 4. Suppose $\{a_n\}$ is a positive sequence such that $\lim_{n\to\infty} a_n = 0$. Show there exists a positive sequence $\{b_n\}$ such that $\lim_{n\to\infty} b_n = 0$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Make sure to rigorously justify your answer by using an ϵ-δ type argument.

Real Analysis, Mandatory Problems

Problem 1. Let $A = [0, 2]$ and $f \in L^1(A)$. Define $f_n : A \to \mathbb{R}$ by

$$f_n(x) = x^{1/n} f(x), \quad n \geq 1$$

Show that $f_n \in L^1(A)$, and find $\lim_{n\to\infty} \int_A f_n$.

Problem 2. Let $A \subset \mathbb{R}^n$ be a set of finite measure, and $f : A \to \mathbb{R}$ be measurable and finite-valued. Suppose that

$$F : A \times A \to \mathbb{R}, \quad F(x, y) = f(x) + f(y)$$

is integrable on $A \times A$. Show that f is integrable on A. Hint: Use the triangle inequality.

Real Analysis, Optional Problems

Problem 3. Let $f : [0, 1] \to (0, \infty)$ be an absolutely continuous function. Show that $\frac{1}{f}$ is also absolutely continuous on $[0, 1]$.

Problem 4. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is measurable, and $\int_E f = 0$ for every measurable set E. Show that $f(x) = 0$ a.e.

Complex Analysis, Mandatory Problems

Problem 1. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function. Assume that there exists a non-negative integer m, such that the inequality $|f(z)| \leq |z|^m$ holds for all $|z| > 2018$. Prove that f is a polynomial of degree at most m.

Problem 2. Use a semicircle contour and the residue theorem to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 10x^2 + 9} dx.$$

Complex Analysis, Optional Problems

Problem 3. Show that the equation $\sin(z) = 2018z^3$ has exactly three solutions inside the unit disk $D(0,1)$.

Problem 4. Recall that for $|\alpha| < 1$, the Möbius transformation

$$w \mapsto \frac{w - \alpha}{1 - \overline{\alpha} w}$$

is a complex diffeomorphism from the open unit disk $D(0,1)$ to itself. Prove that for any analytic function $f\colon D(0,1) \to D(0,1)$ and any $z_0 \in D(0,1)$,

$$\left| \frac{f(z) - f(z_0)}{1 - \overline{f(z_0)} f(z)} \right| \leq \left| \frac{z - z_0}{1 - \overline{z_0} z} \right|.$$