Preliminary Examination in Analysis

May 29, 2019

Instructions

This is a three-hour examination which consists of two parts:

(1) Advanced Calculus
(2) Real or Complex Analysis.

You should work on problems from the section on advanced calculus and from the section of the option you have chosen.

You are to work a total of **five problems**: *four* mandatory problems and *one* optional problem. You must work *two* mandatory problems from each part.

Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Please indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

Problem 1. Suppose that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a (not necessarily complete) metric space X. Show that $d(p_n, q_n)$ has a limit as $n \to \infty$.

Problem 2. Suppose $f : (0,1) \to \mathbb{R}$ has the property that

$$|f(x) - f(y)| \leq |x - y|$$

for all $x, y \in (0,1)$. Show that $\lim_{x \to 0+} f(x)$ exists.

Advanced Calculus, Optional Problems

Problem 3. Suppose $f : [0,1] \to \mathbb{R}$ has the property that

$$|f(x) - f(y)| \leq \frac{1}{2}|x - y|$$

for all $x, y \in [0,1]$. Show that there exists exactly one $x \in [0,1]$ such that $f(x) = x$.

Problem 4. Suppose that $\{f_n\}$ is a decreasing sequence of nonnegative continuous functions on $[0,1]$, i.e., $0 \leq \ldots \leq f_{n+1}(x) \leq f_n(x) \leq \ldots$ for each $x \in [0,1]$. Suppose further that $\lim_{n\to\infty} f_n(x) = 0$ for each x. Show that $f_n \to 0$ uniformly in $[0,1]$.

Real Analysis, Mandatory Problems

Problem 1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable. Show that for any $\varepsilon > 0$, there exists a bounded function g with compact support such that

$$\int_{\mathbb{R}} |f - g| < \varepsilon.$$

Problem 2. Suppose $A \subset \mathbb{R}$ is measurable, and for any open interval (a, b), less than half the points in (a, b) are in A, in the sense that

$$m(A \cap (a, b)) \leq \frac{1}{2}(b - a).$$

Show that A has measure zero.

Real Analysis, Optional Problems

Problem 3. Consider the following statements.

(a): $f \in L^1(\mathbb{R})$,

(b): there exists C such that $m(\{x \in \mathbb{R} : |f(x)| > t\}) \leq Ct^{-1}$ for all $t > 0$.

Give a proof that (a) implies (b), and give an example to show that (b) does not imply (a).

Problem 4. This problem concerns measurable sets and measurable functions.

(a) Suppose that E is a Lebesgue measurable set of measure zero and $B \subset E$. Prove that B has measure zero.

(b) Suppose that E is a Lebesgue measurable set. Suppose that there is a non-negative measurable function f with $f(x) > 0$ a.e. and f is integrable on E. Show that if $\int_E f = 0$, then $m(E) = 0$.

Complex Analysis, Mandatory Problems

Problem 1. Suppose that f is an entire function such that $\lim_{z\to\infty} f(z)$ exists (possibly taking the value ∞). Prove that f is a polynomial.

Problem 2. Use the residue theorem to show that

$$\int_0^\infty \frac{\log x}{x^2+4}\,dx = \frac{\pi}{4}\log 2.$$

Complex Analysis, Optional Problems

Problem 3. Find the number of roots of the equation

$$z^7 - 4z^3 + z - 1 = 0$$

in the open disk $|z| < 1$.

Problem 4. Recall that for $|\alpha| < 1$, the function

$$z \mapsto \frac{z-\alpha}{1-\overline{\alpha}z}$$

is an automorphism of $D(0,1)$ (the open unit disk $|z| < 1$). Let $f\colon D(0,1) \to D(0,1)$ be a holomorphic function with $f(0) = 1/2$. Find a sharp upper bound for $|f'(0)|$, justifying your bound by a proof and its sharpness by an example.

Be sure to justify your answers!